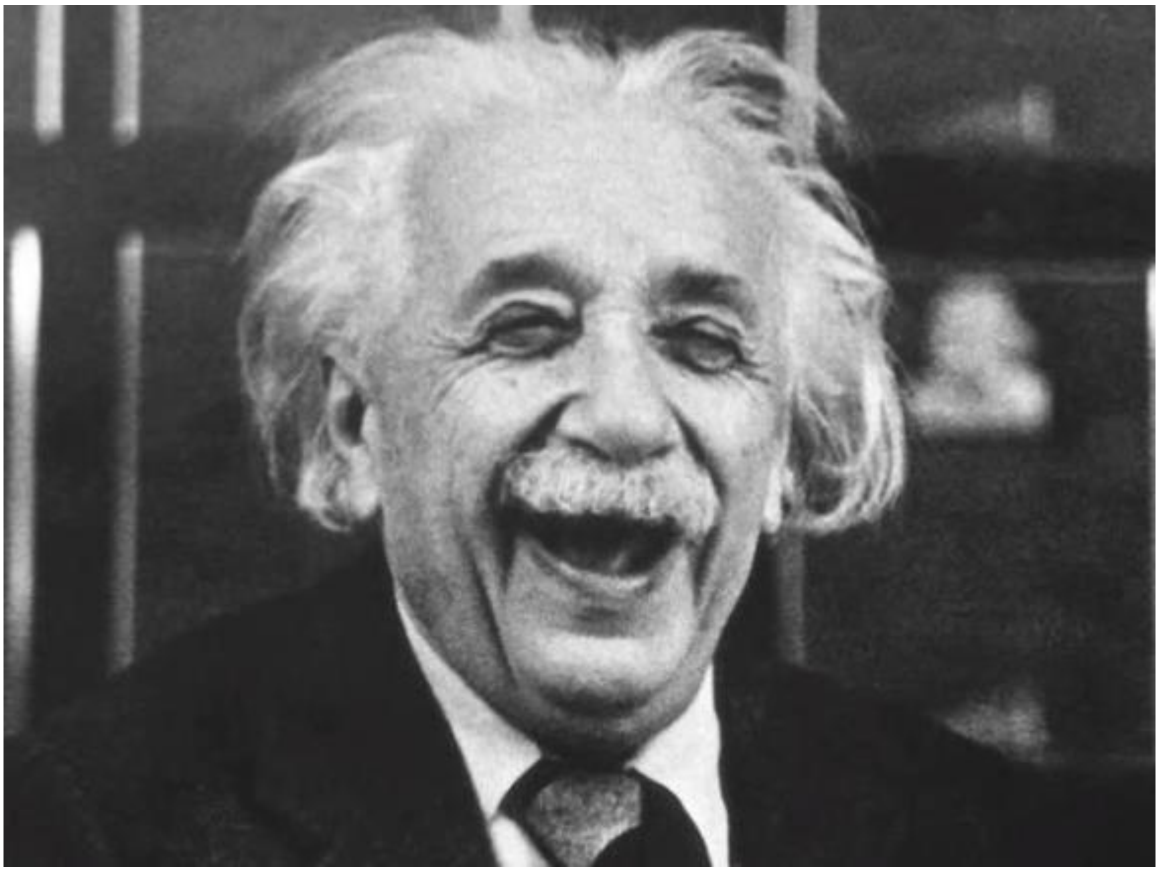
Calculus Option Notes



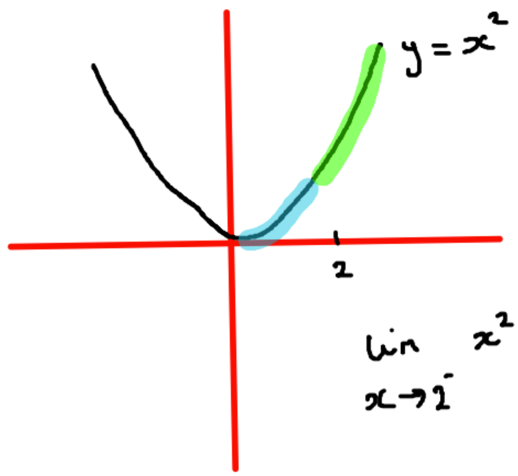
Topic 1:

Essential topics:

1) Limits of sequences – test using L'Hopital or Squeeze
2) Test to see if a function is continuous or differentiable.
3) Rolle and MVT

Does a limit exist?

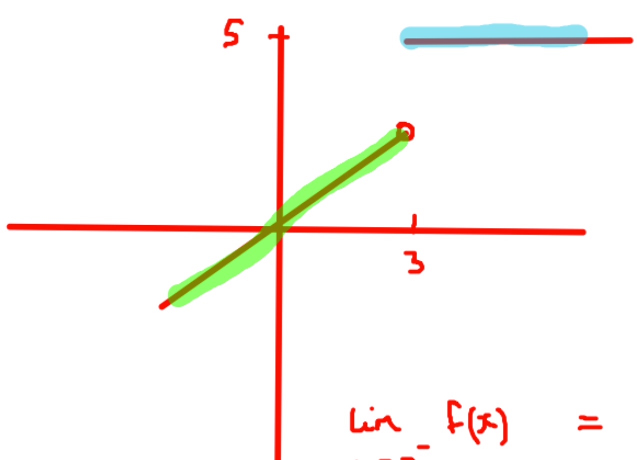
We then say that $\lim_{x\to a} f(x)$ exists and equals l if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l$.



$$\lim_{x\to 2^-} x^2 = 4$$

$$\lim_{x\to 2^+} x^2 = 4$$

$\lim_{x\to 2^-} x^2 = \lim_{x\to 2^+}$ so $\lim_{x\to 2} x^2$ exists



$$f(x) = \begin{cases} x & x < 3 \\ 5 & x \geq 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = 3$$

$$\lim_{x\to 3^+} f(x) = 5$$

So $\lim_{x\to 3}$ does not exist.

Is a function continuous?

A function $f(x)$ is **continuous** at the point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Both the limit and the value $f(x_0)$ must exist for $f(x)$ to be continuous there.

The function is said to be continuous if it is continuous at all points of its domain.

(1) • We then say that $\lim_{x \to a} f(x)$ exists and equals L if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

(2) • and $f(a) = L$

Determine where the following functions are continuous. at $x = 0$

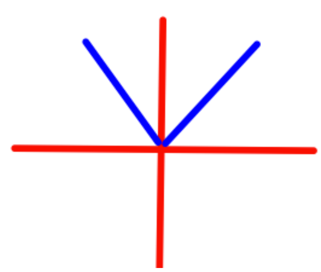
(a) $f(x) = |x|$

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

• $\lim_{x \to 0^-} f(x) = 0$ • $\lim_{x \to 0^+} f(x) = 0$

• $f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$



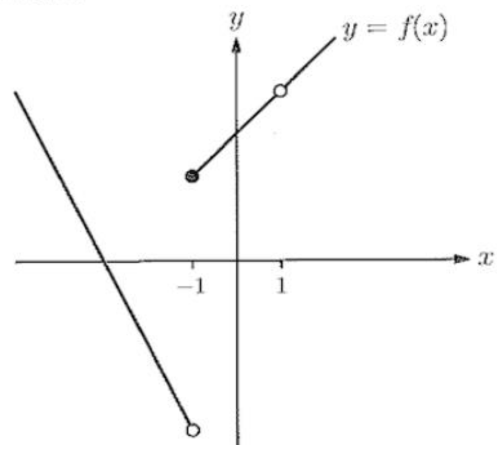
(b) $g(x)=\begin{cases} -2x-6 & x<-1 \\ 2 & x=-1 \\ \dfrac{x^2+2x-3}{x-1} & x>-1, x\neq 1 \end{cases}$

is this continuous ? at $x=1$

(b) $\dfrac{x^2+2x-3}{x-1}=\dfrac{(x+3)(x-1)}{x-1}$

$=x+3 \quad x>-1, x\neq 1$

So, we have:



$\lim_{x\to 1^+} g(x) = 4$

$\lim_{x\to 1^-} g(x) = 4$

But $g(1) \neq 4$

so not continuous

Find b, c such that f(x) is continuous for all x

$f(x) = \begin{cases} 1 & x<3 \\ bx+c & 3\leq x<5 \\ bx^2+2 & x\geq 5 \end{cases}$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$

$1 = 3b + c \quad (1)$

$5b + c = b(25) + 2 \quad (2)$

$1 = 3b + c \quad (1)$

$-2 = 20b - c \quad (2)$

$\therefore$

$-1 = 23b$

$-\frac{1}{23} = b$

$26/23 = c$

Differentiation from first principles

→ The **derivative**, or **gradient function**, of a function f with respect to x is the function $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

differentiate $y = 2x^2$ from first principles

$$\lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$\lim_{h \to 0} \frac{2[x^2 + 2xh + h^2] - 2x^2}{h}$$

$$\lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= 4x$$

Key point: to test if a function is differentiable you must first test that it is continuous.

For a function $f(x)$ to be differentiable at a point x_0:

- $f(x)$ must be continuous at x_0 (and hence $\lim_{x \to x_0} f(x)$ must already exist)
- $f(x)$ must not have a 'sharp point' at x_0
- the tangent to $f(x)$ at x_0 must not be vertical.

If a continuous function $f(x)$ is differentiable at $x = x_0$ then the limits:

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} \text{ and } \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

exist and are equal.

[and not ∞]

Show that: [From first principles]

(a) $f(x)=(x-1)^{\frac{1}{3}}$ is not differentiable at $x=1$.

(a) $$\frac{f(1+h)-f(1)}{h} = \frac{(1+h-1)^{\frac{1}{3}}-(1-1)^{\frac{1}{3}}}{h}$$

$$= \frac{h^{\frac{1}{3}}}{h}$$

$$= \frac{1}{h^{\frac{2}{3}}}$$

$\lim_{h\to 0^+} \frac{1}{h^{2/3}} = \infty.$

$\lim_{h\to 0^-} \frac{1}{h^{2/3}} = \infty$

Show that: [not from first principles]

(a) $f(x)=(x-1)^{\frac{1}{3}}$ is not differentiable at $x=1$.

$f'(x) = \frac{1}{3}(x-1)^{-2/3}$

$\lim_{x\to 1^-} f'(x) = \infty$ $\qquad$ $\lim_{x\to 1^+} f'(x) = \infty$

$f(x)$ not diff at $x=1$

[not from first principles]

$g(x) = |x|$ is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$g'(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

$$\lim_{x \to 0^+} g'(x) = 1 \qquad \lim_{x \to 0^-} g'(x) = -1$$

Find constants a and b so that the function

$$f(x) = \begin{cases} \ln x & x \leq 3 \\ ax + b & x > 3 \end{cases}$$

is differentiable for all $x > 0$.

• First test for continuous:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$$

So

$a(3) + b = \ln 3$ ①

$\Rightarrow 3a + b = \ln 3$

• next differentiate

$$\lim_{x \to 3^-} f'(x) = \lim_{x \to 3^+} f'(x)$$

$$\tfrac{1}{3} = a \quad ②$$

$\therefore$ $a = 1/3$

$b = 1 - \ln 3$

$$w(t)=\begin{cases}2+ct & 0\le t\le 5\\ 16-\dfrac{35}{t} & t>5\end{cases}.$$

(d) Prove from first principles that $w(t)$ is differentiable at $t=5$. *[6]*

(d) differentiable if $\lim_{t\to a^-} f'(t) = \lim_{t\to a^+} f'(t)$

first principles $t=5$ $\quad f'(t) = \lim_{h\to 0} \dfrac{f(t+h)-f(t)}{h}$

$$\lim_{h\to 0^-} \frac{2+\frac{7}{5}(5+h) - \left(2+\frac{7}{5}(5)\right)}{h} = \frac{\frac{7}{5}h}{h}$$

$$\lim_{h\to 0^-} = 7/5$$

replace t with 5+h

$$\lim_{h\to 0^+} \frac{\left(16-\frac{35}{5+h}\right)-\left(16-\frac{35}{5}\right)}{h}$$

$$\lim_{h\to 0^+} = \frac{\frac{-35}{5+h}+7}{h} = \frac{-\frac{35}{5+h}+\frac{7(5+h)}{5+h}}{h}$$

$$\lim_{h\to 0^+} = \frac{\frac{-35+35+7h}{5+h}}{h} = \frac{7}{5+h} = \frac{7}{5}$$

Sequences

$$\lim_{n\to\infty}\frac{n^2+5}{2n^2-3n+8}=\lim_{n\to\infty}\frac{1+\frac{5}{n^2}}{2-\frac{3}{n}+\frac{8}{n^2}}$$

$$=\frac{1+0}{2-0+0}$$

The sequence $\{u_n\}$ is defined by $u_n=\frac{3n+2}{2n-1}$, for $n\in\mathbb{Z}^+$.

a. Show that the sequence converges to a limit L, the value of which should be stated. $L = 3/2$

b. Find the least value of the integer N such that $|u_n - L| < \varepsilon$, for all $n > N$ where

(i) $\varepsilon = 0.1$;
(ii) $\varepsilon = 0.00001$.

$|u_n - L| < 0.1$

$\left|\frac{3n+2}{2n-1}-\frac{3}{2}\right| < 0.1$

$\left|\frac{7}{2(2n-1)}\right| < 0.1$

$7 < 0.1(4n-2)$

$72/4 < n$

$18 < n$

so $N = 18$

$\left|\frac{7}{2(2n-1)}\right| < 0.00001$

$7 < 0.00001(4n-2)$

$\frac{702{,}000}{4} < n$

$175{,}000 < n$

$N = 175{,}000$

$u_n = \frac{3n+2}{2n-1}$;

For each of the sequences $\left\{\frac{u_n}{n}\right\}$, $\left\{\frac{1}{2u_n-2}\right\}$ and $\{(-1)^n u_n\}$, determine whether or not it converges.

(a) $\lim\limits_{n\to\infty} u_n = \frac{3}{2}$

and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

so $\lim\limits_{n\to\infty} \frac{u_n}{n} = \lim\limits_{n\to\infty} \frac{3/2}{n} = 0$ converges ✓

(b) $\lim\limits_{n\to\infty} \frac{1}{2u_n-2} = \frac{1}{2(3/2)-2} = 1$ converges ✓

(c) $\lim\limits_{n\to\infty} (-1)^n u_n$ will alternate between values close to $\pm 3/2$ so does not converge.

Let $g(x) = \sin x^2$, where $x \in \mathbb{R}$.

a. Using the result $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$, or otherwise, calculate $\lim\limits_{x\to 0} \frac{g(2x)-g(3x)}{4x^2}$.

$$\lim_{x\to 0} \frac{\sin 4x^2 - \sin 9x^2}{4x^2}$$

$$\lim_{x\to 0} \frac{\sin 4x^2}{4x^2} - \frac{\sin 9x^2}{4x^2}$$

$$\lim_{x\to 0}\left(\frac{\sin 4x^2}{4x^2}\right) - \frac{9}{4}\lim_{x\to 0}\left(\frac{\sin 9x^2}{9x^2}\right)$$

set $u = 4x^2$ as $x \to 0$ $u \to 0$ set $v = 9x^2$ as $x\to 0$ $v \to 0$

$$\lim_{u\to 0}\left(\frac{\sin u}{u}\right) - \frac{9}{4}\lim_{v\to 0}\left(\frac{\sin v}{v}\right)$$

$$1 - \tfrac{9}{4}(1) = -\tfrac{5}{4}$$

L'Hopital's Rule

INDETERMINATE FORMS

The theorems for limits of functions above do not help us to deal with *indeterminate forms*. These include:

Type	Description
$\frac{0}{0}$	$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$
$\frac{\infty}{\infty}$	$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ where $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$
$0 \times \infty$	$\lim\limits_{x\to a} [f(x)g(x)]$ where $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \pm\infty$

L'HÔPITAL'S RULE

Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ on an interval that contains a point $x = a$.

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, or, if $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$,

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ provided the limit on the right exists.

Use L'Hôpital's Rule to evaluate: **a** $\lim\limits_{x\to 0} \frac{2^x - 1}{x}$ **b** $\lim\limits_{x\to 0} \frac{\sin x}{x}$.

a $\lim\limits_{x\to 0} (2^x - 1) = 0$ and $\lim\limits_{x\to 0} x = 0$, so we can use L'Hôpital's Rule.

$$\therefore \lim_{x\to 0} \frac{2^x - 1}{x} = \frac{\lim\limits_{x\to 0} \frac{d}{dx}(2^x - 1)}{\lim\limits_{x\to 0} \frac{d}{dx}(x)} \quad \{\text{L'Hôpital's Rule}\}$$

$$= \frac{\lim\limits_{x\to 0} 2^x \ln 2}{\lim\limits_{x\to 0} 1}$$

$$= \frac{\ln 2}{1} = \ln 2$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\lim_{x\to\infty} \frac{d}{dx}(\ln x)}{\lim_{x\to\infty} \frac{d}{dx}(x)} \quad \{\text{L'Hôpital's Rule}\}$$

$$= \frac{\lim_{x\to\infty} \left(\frac{1}{x}\right)}{\lim_{x\to\infty} 1}$$

$$= \frac{0}{1} \quad \{\text{since} \quad \lim_{x\to\infty} \left(\frac{1}{x}\right) = 0\}$$

$$= 0$$

Sometimes you use L'Hopital's rule and still end up with either 0/0 or inf/inf.
In this case you can use the rule one more time.

(b) $\lim_{x\to 1} \dfrac{1 - x^2 + 2x^2 \ln x}{1 - \sin \dfrac{\pi x}{2}}$.

(7)
(Total 11 marks)

$\frac{0}{0}$ ✓

$$= \lim_{x\to 1} \frac{-2x + 2x^2\left(\frac{1}{x}\right) + 4x\ln x}{-\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)} = \frac{4x\ln x}{-\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)}$$

But still $\frac{0}{0}$!

$$= \frac{4\ln x + 4x\left(\frac{1}{x}\right)}{\frac{\pi^2}{4}\sin\frac{\pi x}{2}} \quad \text{and} \quad \lim_{x\to 1} = \frac{16}{\pi^2}$$

L'Hôpital's rule can also be used to find limits of the form '$0 \times \infty$' or '$\infty - \infty$'. First it is necessary to rearrange these expressions into a quotient which is the of the form $\frac{'0'}{0}$ or $\frac{'\infty'}{\infty}$.

Using l'Hopital's Rule, show that $\lim_{x \to \infty} xe^{-x} = 0$.

rewrite $\lim_{x \to \infty} \frac{x}{e^x}$ $\quad \frac{\infty}{\infty}$ ✓

$= \lim_{x \to \infty} \frac{1}{e^x} = 0$

$$\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \left(\frac{\ln x}{\frac{1}{x}} \right) \\
&= \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}} \right) \quad \{\text{L'Hôpital's Rule}\} \\
&= \lim_{x \to 0^+} (-x) \\
&= 0
\end{aligned}$$

Calculate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

rewrite as: $\lim_{x \to 0} \frac{\sin x - x}{x \sin x}$ $\frac{0}{0}$ ✓

∴ L'hopital again $\lim_{x \to 0} \frac{\cos x - 1}{x \cos x + \sin x}$ $\frac{0}{0}$ ✓

$\lim_{x \to 0} \frac{-\sin x}{-x \sin x + \cos x + \cos x} = 0$

Squeeze Theorem

Squeeze Theorem

If we have sequences $\{a_n\},\{b_n\}$ and $\{c_n\}$ such that

$a_n \le b_n \le c_n$ for all $n \in \mathbb{Z}^+$

and

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L < \infty$$

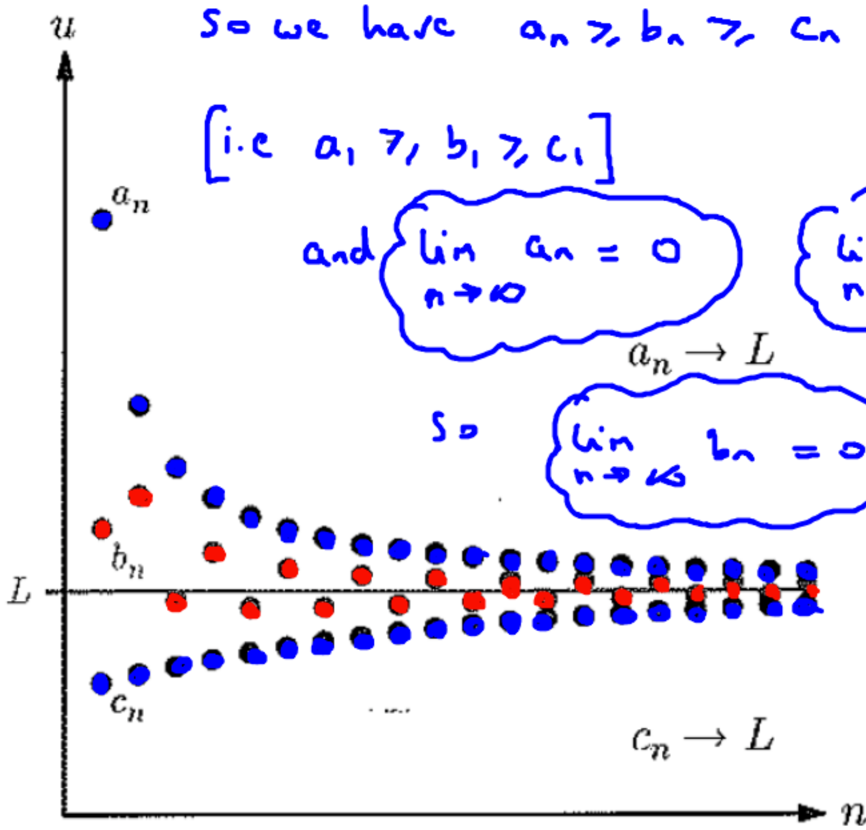
then $\lim_{n \to \infty} b_n = L$.

So we have $a_n \geq b_n \geq c_n$ for all $n \in \mathbb{Z}$

[i.e $a_1 \geq b_1 \geq c_1$]

and $\lim_{n\to\infty} a_n = 0$ $\quad \lim_{n\to\infty} c_n = 0$

so $\lim_{n\to\infty} b_n = 0$



Use the Squeeze Theorem to find $\lim\limits_{n\to\infty} \dfrac{\sin n}{n}$.

$-1 \leq \sin n \leq 1$ for all $n \in \mathbb{Z}^+$

$\Rightarrow \dfrac{-1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$ for all $n \in \mathbb{Z}^+$

[look for upper + lower bounds]

Since

$\lim\limits_{n\to\infty}\left(-\dfrac{1}{n}\right) = \lim\limits_{n\to\infty}\dfrac{1}{n} = 0$

by the Squeeze Theorem

$\lim\limits_{n\to\infty}\dfrac{\sin n}{n} = 0$

Show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Next,

$$\frac{n!}{n^n} = \frac{n}{n}\frac{(n-1)}{n}\frac{(n-2)}{n}\frac{(n-3)}{n}\cdots\frac{3}{n}\frac{2}{n}\frac{1}{n}$$

$$< \frac{n}{n}\frac{n}{n}\frac{n}{n}\frac{n}{n}\cdots\frac{n}{n}\frac{n}{n}\frac{1}{n} = \frac{1}{n}$$

$$0 < \frac{n!}{n^n} < \frac{1}{n} \quad \text{for all } n \in \mathbb{Z}^+$$

$\lim_{n\to\infty} \frac{1}{n} = 0$ So by squeeze theorem

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

$$u_n = \frac{3n + \sin(2n)}{4n-3} \quad n \in \mathbb{Z}^+$$

Use squeeze theorem to find $\lim_{n\to\infty} u_n$

$$\frac{3n-1}{4n-3} \leq \frac{3n+\sin(2n)}{4n-3} \leq \frac{3n+1}{4n-3}$$

and $\lim_{n\to\infty} \frac{3n+1}{4n-3} = \lim_{n\to\infty} \frac{3+1/n}{4-3/n} = 3/4$

$$\lim_{n\to\infty} \frac{3n-1}{4n-3} = 3/4$$

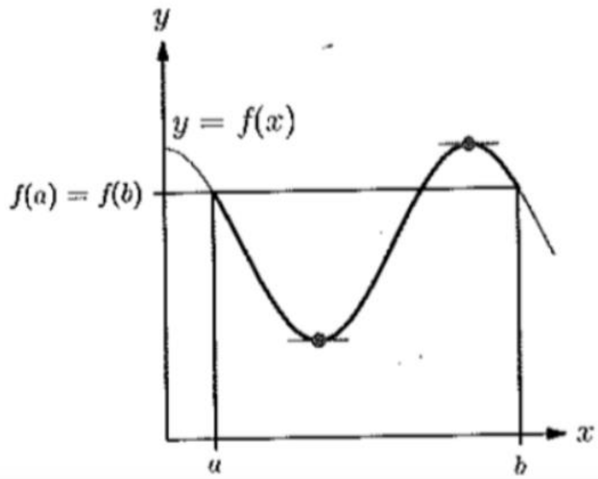
$\therefore \lim_{n\to\infty} u_n = 3/4$ ✓

Rolles Theorem and MVT

Rolle's Theorem

For a function, $f(x)$, that is continuous on an interval $[a,b]$ and differentiable on $]a,b[$,

if $f(a)=f(b)$ then there must exist a point $c \in]a,b[$ such that $f'(c)=0$.



$f(x) = \cos 2x + 2\cos x \quad 0 \leq x \leq 2\pi$

use Rolle's theorem to show $f'(x)$ has at least 1 solution on $]0, 2\pi[$. Hence find all sols

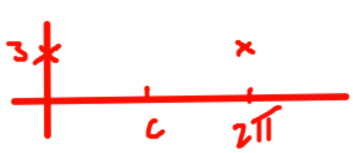
$f(x)$ is continuous on $0 \leq x \leq 2\pi$ ✓

differentiable on $0 < x < 2\pi$ ✓

if $f(a)=f(b)$ then there must exist a point $c \in]a,b[$ such that $f'(c)=0$.

$f(0) = 3 \quad f(2\pi) = 3 \quad$ ie $f(0) = f(2\pi)$ ✓

$\therefore$ must have a point c $0 < c < 2\pi$ with $f'(c) = 0$



$f(x) = \cos 2x + 2\cos x \quad 0 \leq x \leq 2\pi$

So we find c that makes $f'(c) = 0$

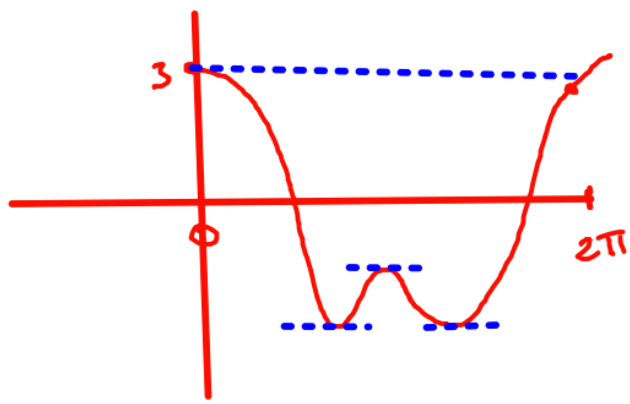
$f'(x) = -2\sin 2x + -2\sin x = 0$

$2\sin x = -2\sin 2x$

$2\sin x = -2(2\sin x \cos x)$

either $\sin x = 0$ or $\cos x = -\frac{1}{2}$

$x = \pi \qquad x = \frac{2\pi}{3} \qquad x = \frac{4\pi}{3}$



What did we need Rolle's theorem for?

to prove there was at least 1 sol to $f'(x) = 0$ between 0 and 2π.

The most common application of Rolle's Theorem is to establish a maximum number of possible roots of a polynomial.

Prove that the polynomial $f(x) = x^3 + 3x^2 + 6x + 1$ has exactly one root.

$f(x)$ is continuous for all $x \in \mathbb{R}^+$ ✓

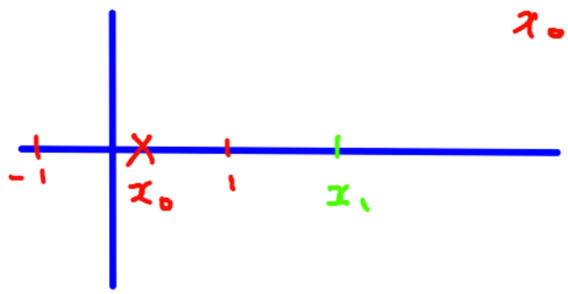
$f(x)$ is differentiable for all $x \in \mathbb{R}$ ✓

step (1) show 1 root exists.

$f(-1) = -3$ $\qquad$ $f(1) = 11$

therefore as $f(x)$ continuous we have at least 1 root between $-1 < x < 1$

Step (2) $\qquad$ x_0 first root



look for contradiction. Say there was a 2nd root, x_1

$x_1 > x_0$

Now we could use Rolle's theorem

i.e we would have $f(x_0) = f(x_1)$ and would need a point between x_0 and x_1 with gradient 0.

But $f'(x) = 3x^2 + 6x + 6 \neq 0$ $\qquad$ $3[(x+1)^2 + 3]$

which is contradiction $\rightarrow$ only 1 root x_0

The function f is defined by $f(x)=\begin{cases} e^{-x^2}\left(-x^3+2x^2+x\right), & x\le 1 \\ ax+b, & x>1 \end{cases}$, where a and b are constants.

(a) Find the exact values of a and b if f is continuous and differentiable at $x=1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\lim_{x\to 1^-} e^{-x^2}\left(-x^3+2x^2+x\right) = \lim_{x\to 1^+} ax+b$$

$$2e^{-1} = a+b$$

next

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$$

$$\lim_{x\to 1^-} e^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right) = \lim_{x\to 1^+} a$$

$\therefore -2e^{-1} = a$ (2)

now sub into (1) to find $b = 4e^{-1}$.

(b) (i) Use Rolle's theorem, applied to f, to prove that $2x^4-4x^3-5x^2+4x+1=0$ has a root in the interval $]-1,1[$.

(ii) Hence prove that $2x^4-4x^3-5x^2+4x+1=0$ has at least two roots in the interval $]-1,1[$.

from (a) we have

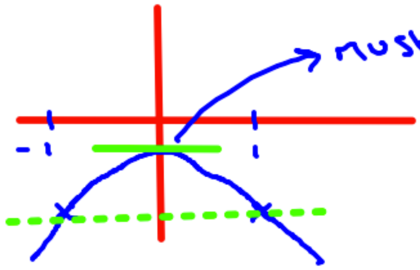
$$f'(x) = e^{-x^2}\left(2x^4-4x^3-5x^2+4x+1\right)$$

as $e^{-x^2} \neq 0$ then $f'(x)=0 \Rightarrow 2x^4-4x^3-5x^2+4x+1=0$

use Rolle's theorem

$f(-1) = -2$ and $f(1) = -2$ use boundaries

$\therefore$ must have $f'(x) = 0$ between $x \in]-1, 1[$



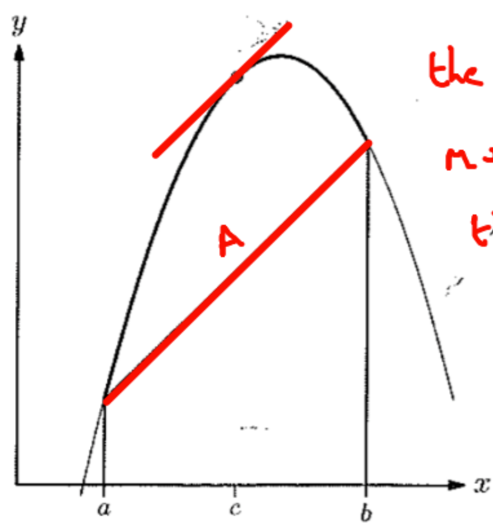
$f'(x) = 0 \Rightarrow \underbrace{2x^4 - 4x^3 - 5x^2 + 4x + 1 = 0}$

So this has a root between $x \in]-1, 1[$

MVT

Mean Value Theorem → generalised version of Rolle

For a function, $f(x)$, that is continuous on an interval $[a,b]$ and differentiable on $]a;b[$, there must exist a point $c \in]a,b[$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.



the gradient of line A must be same as that of some other tangent in the interval.

The mean value theorem states that if f is a continuous function on $[a, b]$ and differentiable on $]a, b[$ then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in]a, b[$.

(i) Find the two possible values of c for the function defined by $f(x) = x^3 + 3x^2 - 2$ on the interval $[-3, 1]$.

(ii) Illustrate this result graphically. [7]

LHS

$f'(x) = 3x^2 + 6x$

$f'(c) = 3c^2 + 6c$

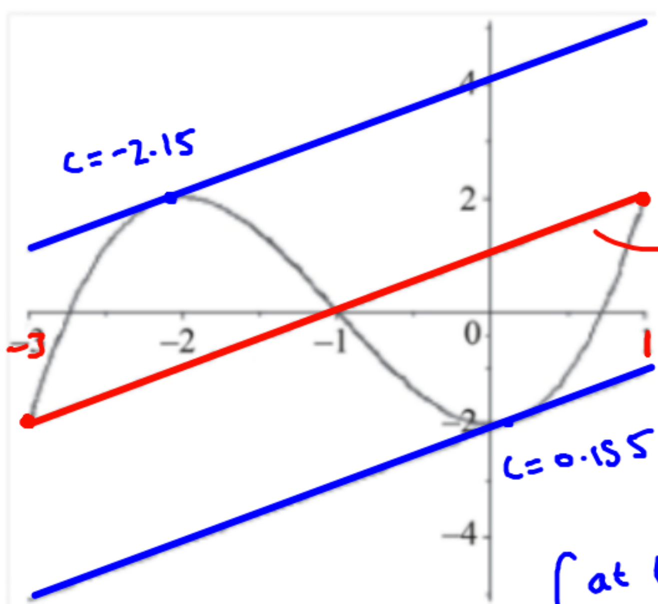
RHS

$\dfrac{f(1) - f(-3)}{1 - -3} = \dfrac{2 - -2}{4}$

$\therefore \quad 3c^2 + 6c = 1$

$3c^2 + 6c - 1 = 0 \qquad c = -2.15 \qquad c = 0.155$

(ii)



this has gradient 1

[at these c values we have same gradient]

Prove that $|\sin a - \sin b| \leq |a - b|$.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

set $f(x) = \sin x$

$$\Rightarrow \frac{\sin b - \sin a}{b - a} = \cos c$$

note!

$$\Rightarrow \left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \leq 1$$

$|a-b| = |b-a|$

$$\Rightarrow |\sin a - \sin b| \leq |a - b|$$

If $f(x)$ is such that $f(2) = -4$ and $f'(x) \geq -2$ for all $x \in]2,7[$, find the smallest possible value for $f(7)$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$a < c < b$

cont ✓ diff ✓

$$-2 \leq f'(c) = \frac{f(7) - f(2)}{7 - 2}$$

$$-2 \leq \frac{f(7) - f(2)}{5}$$

$$-10 \leq f(7) - -4$$

$$-14 \leq f(7)$$

• Prove that $e^x > 1+x$

• Write $f(x) = e^x - 1 - x$

• MVT $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ for $[a,b]$

• take $a=0$, $b=x \to$ variable

$$\frac{f(x)-f(0)}{x} = e^c - 1 > 0 \quad c \in]0,x[$$

$f(x) > 0$

$e^x - 1 - x > 0 \quad \rightarrow \quad e^x > 1+x$ ✓

The function f is continuous on $[a,b]$, differentiable on $]a,b[$ and $f'(x)=0$ for all $x \in]a,b[$. Show that $f(x)$ is constant on $[a,b]$.

Hence, prove that for $x \in [0,1]$, $2\arccos x + \arccos(1-2x^2) = \pi$.

need to test any arbitrary interval inside $[a,b]$

Say we have 2 values $x_1, x_2 \in]a,b[$ then

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \quad \text{for some } c \in]x_1, x_2[$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$$

and as x_1, x_2 are arbitrary

$\therefore f(x_2) = f(x_1) \rightarrow f(x)$ constant for $[a,b]$

Hence, prove that for $x \in [0, 1]$, $2\arccos x + \arccos(1-2x^2) = \pi$.

MVT : $\dfrac{f(b)-f(a)}{b-a} = f'(c)$. Take $b = x$, $a = 0$

MVT : $\dfrac{f(x)-f(0)}{x-0} = f'(c)$ where $x \in [0,1]$

Set $f(x) = 2\arccos x + \arccos(1-2x^2)$

RHS: $f'(x) = -2\dfrac{1}{\sqrt{1-x^2}} - \dfrac{-4x}{\sqrt{1-(1-2x^2)^2}}$

$= -2\dfrac{1}{\sqrt{1-x^2}} + \dfrac{4x}{\sqrt{4x^2-4x^4}}$

use formula book.

$\arccos u \to \dfrac{1}{\sqrt{1-u^2}}$

RHS $f'(x) = -2\dfrac{1}{\sqrt{1-x^2}} + \dfrac{4x}{\sqrt{4x^2-4x^4}}$

$= -2\dfrac{1}{\sqrt{1-x^2}} + \dfrac{4x}{2x}\dfrac{1}{\sqrt{1-x^2}}$

$= 0$

But if $f'(x) = 0$ for all $x \in [0,1]$

then as we have

$\dfrac{f(x)-f(0)}{x-0} = f'(c)$ then $f(x) - f(0) = 0$

$f(x) = f(0)$

and $f(0) = 2\arccos 0 + \arccos(1-2(0)^2) = \pi$

$\therefore f(x) = \pi$ as required ✓

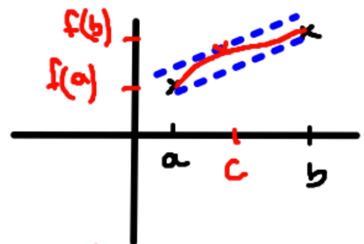
f is a continuous function defined on $[a, b]$ and differentiable on $]a, b[$ with $f'(x) > 0$ on $]a, b[$.

$x_1 \quad x_2 \qquad x_2 > x_1$

Use the mean value theorem to prove that for any $x, y \in [a, b]$, if $y > x$ then $f(y) > f(x)$. *[4]*

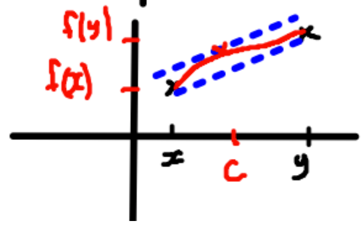
MVT have $x \in \,]a,b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$



So our domain is $x, y \in [a,b]$

and our point is c. $\therefore$



$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

$$f'(c) = \frac{f(y) - f(x)}{y - x} > 0 \qquad \text{as } f'(x) > 0$$

But $y > x$ so $y - x > 0$

$\therefore \quad f(y) - f(x) > 0$

(d) (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$.

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

(i) $g(x) = x - \arctan x$

$$g'(x) = 1 - \frac{1}{1 + x^2} \qquad \text{*use formula book*}$$

$\therefore$ as $\dfrac{1}{1+x^2} < 1$ for $x > 0$

$g'(x) > 0$

Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

(ii)

$g(x) = x - \arctan x$

$\therefore$ from (c) for any $x_1\ x_2 \in [a,b]$ $\quad x_2 > x$

$g(x_2) > g(x_1)$

if choose $x_1 = 0$

$g(x_2) > g(0) = 0 \qquad$ for all $x_2 > 0$

ie $\quad g(x) = x - \arctan x > 0$

$\qquad\qquad x > \arctan x$

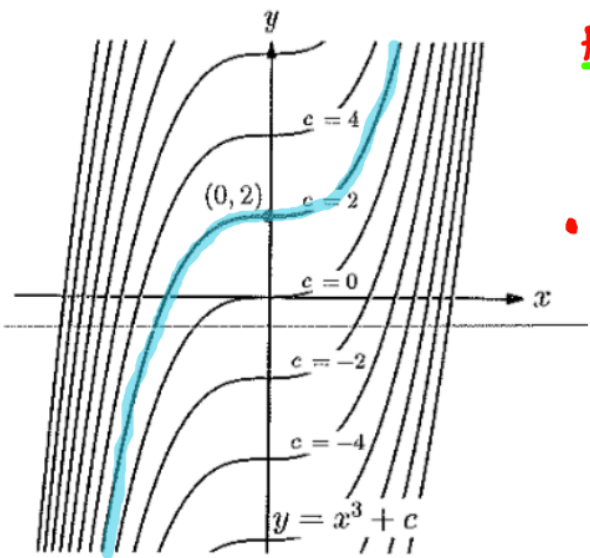
Topic 2

Essential Topics:

Solving differential equations by separate variables.
Solving differential equations by substitution.
Solving differential equations by integrating factor
Solving differential equations by Euler's Method
Sketching slope fields and isoclines

So general solution to $\frac{dy}{dx} = 3x^2$

$y = x^3 + c$



Particular solution needs boundary conditions.

• eg. solve $\frac{dy}{dx} = 3x^2$ given solution passes through $(2, 0)$

$y = x^3 + c$

$2 = 0 + c \rightarrow y = x^3 + 2$

5B Separation of variables

The second type of differential equation which you need to be able to solve is one that can be written in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

eg $\frac{dy}{dx} = \frac{x}{y}$

$y\,dy = x\,dx$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c \rightarrow y = \pm\sqrt{x^2+2c}$

Show that the general solution to the differential equation

$$\frac{dy}{dx} = xy - x$$

can be written as $y = 1 + Ae^{x^2}$ if $y > 1$.

$$\frac{dy}{dx} = x(y-1)$$

$$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$$\ln|y-1| = \frac{x^2}{2} + c$$

$$\Rightarrow \quad |y-1| = e^{x^2+c}$$

But since $y-1>0$

$$y-1 = e^{x^2+c}$$
$$= e^{x^2}e^{c}$$

$y = Ae^{x^2} + 1 \qquad A = e^{c}$

Substitution

KEY POINT 5.3

Any homogeneous differential equation can be converted to a variables separable differential equation (if it is not already) by making the change of variable (or substitution):

$$y = vx$$

Here v is a variable and not a constant, so in making this substitution we must be sure to differentiate the product when replacing $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(vx)\\ &= x\frac{dv}{dx} + (v \times 1)\\ &= x\frac{dv}{dx} + v\end{aligned}$$

A homogeneous differential equation is one of the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

For example $\frac{dy}{dx} = \frac{y^2}{x^2}$ and $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ are both homogeneous

because

$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$ and $\frac{dy}{dx} = \frac{1}{2}\left(\frac{x^2}{xy} + \frac{y^2}{xy}\right) = \frac{1}{2}\left(\frac{1}{\left(\frac{y}{x}\right)} + \frac{y}{x}\right)$ respectively.

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + y/x$$

homogenous because $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ ✓

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}x + v$ [product rule]

$\frac{dv}{dx}x + v = (v)^2 + v$ as $\frac{y}{x} = v$

$\frac{dv}{dx}x = v^2$

$$x\frac{dv}{dx} = v^2$$

$$\frac{dv}{v^2} = \frac{dx}{x}$$

$$-v^{-1} = \ln x + c$$

$$-\left(\frac{y}{x}\right)^{-1} = \ln x + c$$

$$-\frac{x}{y} = \ln x + c$$

$$\frac{-x}{\ln x + c} = y$$

Find the general solution of $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ $x, y > 0$ in the form $y^2 = f(x)$.

Let $y = vx$

Then, $\frac{dy}{dx} = \frac{d}{dx}(vx)$

$= x\frac{dv}{dx} + v$ and so,

$$x\frac{dv}{dx} + v = \frac{x^2 + (vx)^2}{2x(vx)}$$

$$\Rightarrow x\frac{dv}{dx} + v = \frac{x^2 + v^2x^2}{2vx^2}$$

$$\Rightarrow x\frac{dv}{dx} + v = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\ln|1-v^2| = \ln x + C$$

$$\Rightarrow \ln|1-v^2| = \ln\tfrac{1}{x} - C$$

$$\Rightarrow 1-v^2 = e^{\ln\frac{1}{x} - C}$$

$$1-v^2 = e^{\ln\frac{1}{x}} e^{-c}$$

$$1-v^2 = \frac{A}{x}$$

$$1 - v^2 = \frac{A}{x}$$

$$\therefore 1 - \frac{y^2}{x^2} = \frac{A}{x}$$

$$\Rightarrow \quad x^2 - y^2 = Ax$$

$$\Rightarrow \quad y^2 = x(x - A)$$

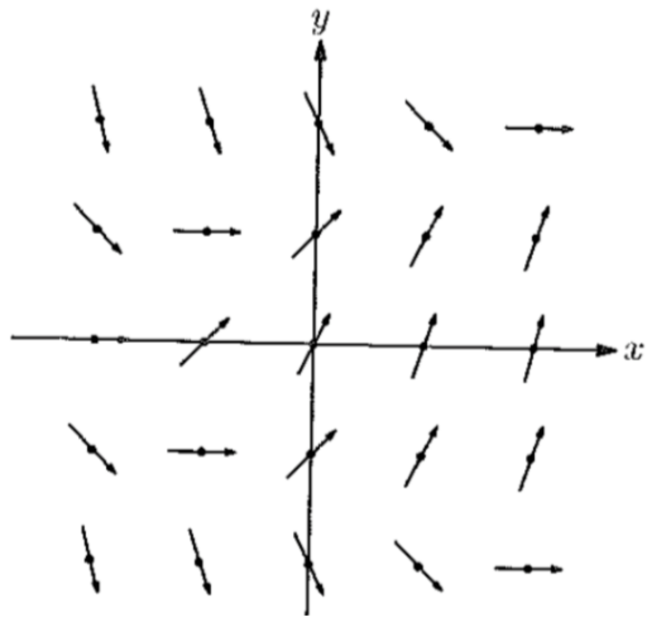
Slope fields

$$\frac{dy}{dx} = x - y^2 + 2$$

Continuing this process for a range of coordinates, we can build up a table showing the gradient at various points:

		x				
		−2	−1	0	1	2
y	−2	−4	−3	−2	−1	0
	−1	−1	0	1	2	3
	0	0	1	2	3	4
	1	−1	0	1	2	3
	2	−4	−3	−2	−1	0

And from here we can represent the gradient at each point graphically by drawing the tangent at that point:

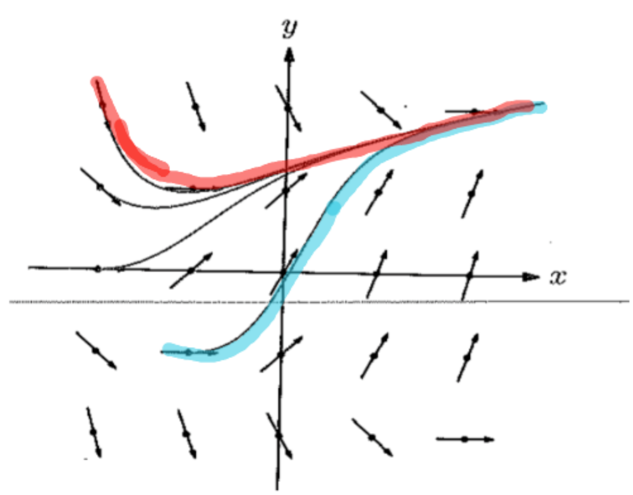


A plot of the tangents at all points (x, y) is called the **slope field** of a differential equation.

From the slope field, we can then construct approximate solution curves that correspond to different initial conditions. To do so we just observe two rules.

Solution curves:

1. follow the direction of the tangents at each point
2. do not cross.



Isoclines

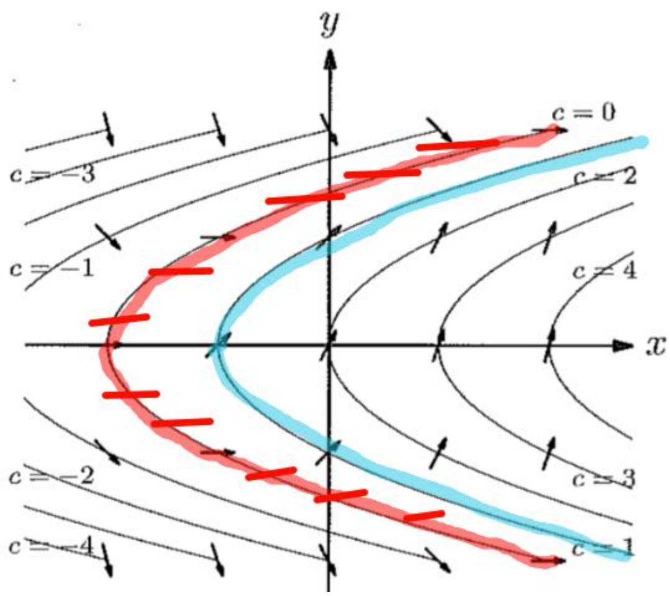
A curve on which all points have the same gradient is known as an **isocline**.
To find isoclines set $\frac{dy}{dx} = c$ for some constant c.

In the example above, with $\frac{dy}{dx} = x - y^2 + 2$ the isoclines will be given by:

$c = x - y^2 + 2 \Rightarrow y^2 = x + 2 - c.$

Therefore, on the isocline corresponding to:

- $c = 0$ $(y^2 = x + 2)$, the tangents at every point will have gradient 0
- $c = 1$ $(y^2 = x + 1)$, the tangents at every point will have gradient 1 and so on.



Note these lines are not solutions to $\frac{dy}{dx}$!

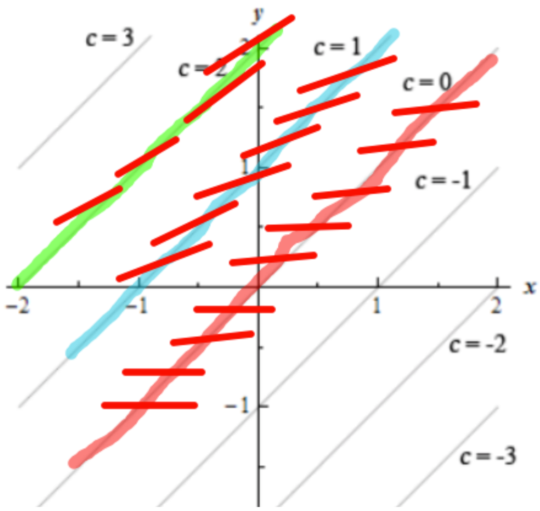
sketch isoclines for
$\frac{dy}{dx} = y - x$

set $c = 0, 1, 2$

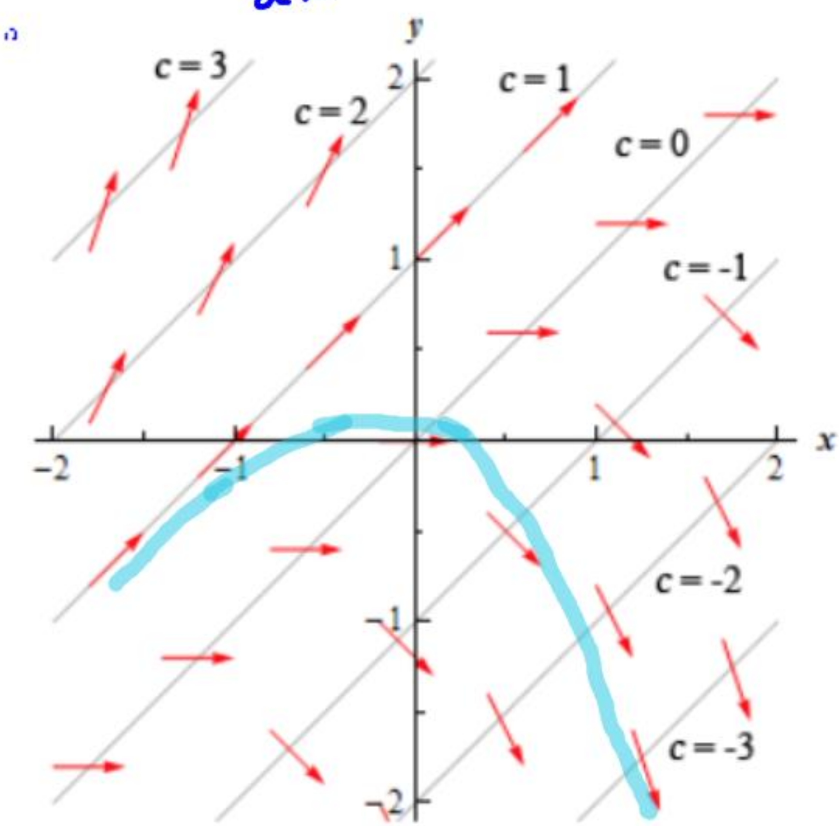
$c = 0 \rightarrow 0 = y - x$
$y = x$

$c = 1 \rightarrow 1 = y - x$
$y = x + 1$

$c = 2 \rightarrow 2 = y - x$
$y = x + 2$



Sketch a particular solution to
$\frac{dy}{dx} = y - x$ through $(0,0)$



Euler's Method

Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant (step length)

Non-Calc

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ where $f(x, y) = y - 2x$. C through (0,1)

(d) Use Euler's method with a step interval of 0.1 to find an approximate value for y on C, when $x = 0.5$. *[4]*

n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + h \times f(x_n, y_n)$
0	0	1	$1 - 0 = 1$	$1 + 0.1(1) = 1.1$
1	0.1	1.1	$1.1 - 2(0.1) = 0.9$	$1.1 + 0.1(0.9) = 1.19$
2	0.2	1.19	$1.19 - 2(0.2) = 0.79$	$1.19 + 0.1(0.79) = 1.269$
3	0.3	1.269	$1.269 - 2(0.3) = 0.669$	$1.269 + 0.1(0.669) = 1.3359$
4	0.4	1.3359	$1.3359 - 2(0.4) = 0.5359$	$1.3359 + 0.1(0.5359) = 1.38949$ 1.39 (3sf).

Calculator method

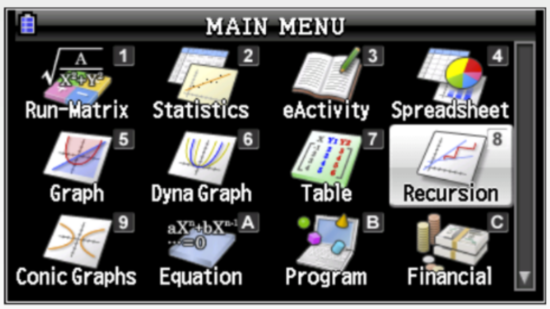
Given that $\frac{dy}{dx} = x + 2y$, and $y = 0$ when $x = 1$, use Euler's method with a step value of 0.1 to approximate y when $x = 1.3$.

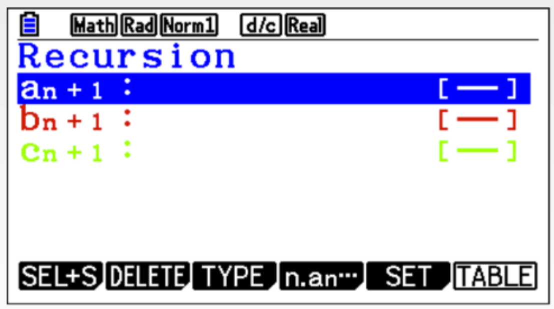
On paper, set up the following table: [remember that $F(x, y) = \frac{dy}{dx}$]

x	y	f(x,n) $\frac{dy}{dx}$	h× f(x,n) $\delta y = h \times \frac{dy}{dx}$
1	0		
1.1			
1.2			
1.3			

(you can pre-fill the x values as you know the step size, and when to stop)

On the calculator… select *Recursion*

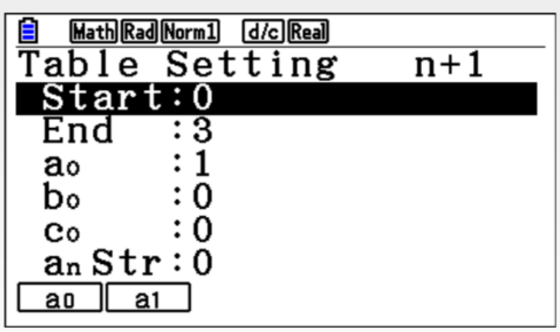


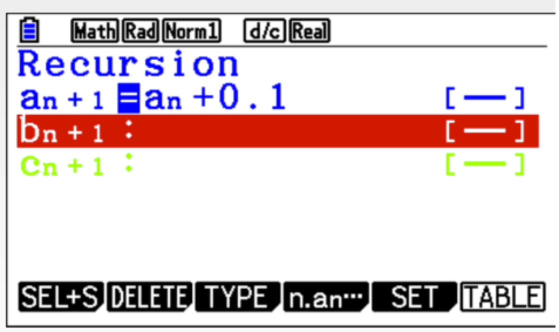


Set up the problem by letting $x_n \equiv a_n$, $y_n \equiv b_n$ and $\frac{dy}{dx} \equiv c_n$

F5 – SET

$$x_{n+1} = x_n + h$$



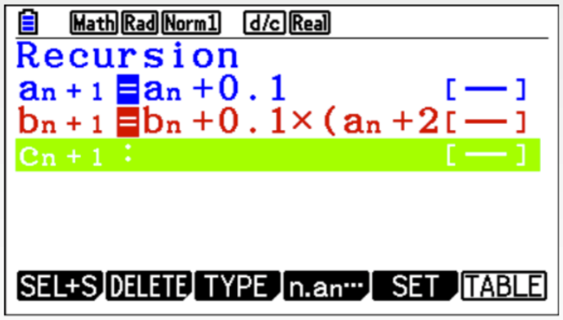


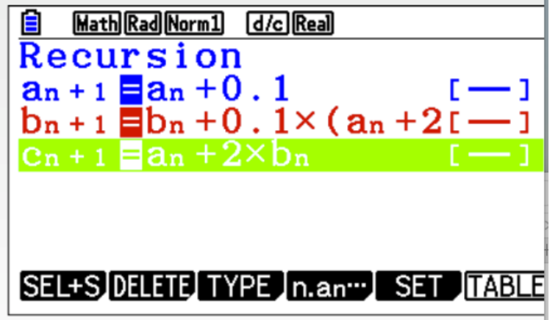
- End = number of iterations

$$a_{n+1} = a_n + 0.1$$

- $(x_0, y_0) \equiv (a_0, b_0)$

$$y_{n+1} = y_n + h \times \frac{dy}{dx} \quad \text{where } \frac{dy}{dx} = x + 2y \qquad \frac{dy}{dx} = F(x, y) = x + 2y$$

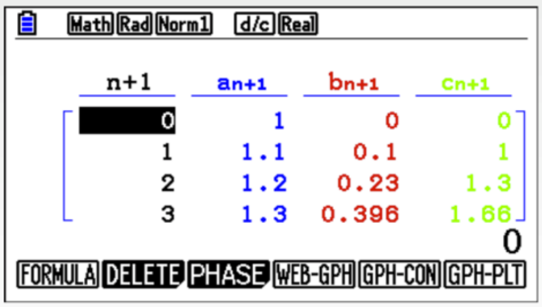


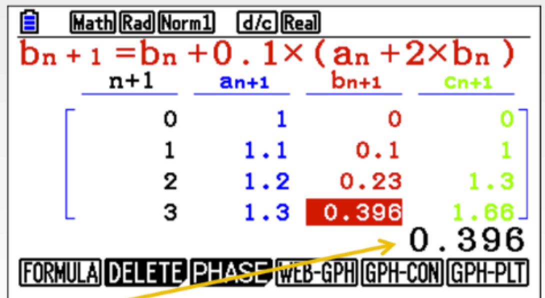


$$b_{n+1} = b_n + 0.1 \times (a_n + 2b_n) \qquad c_{n+1} = a_n + 2b_n$$

F6 – TABLE

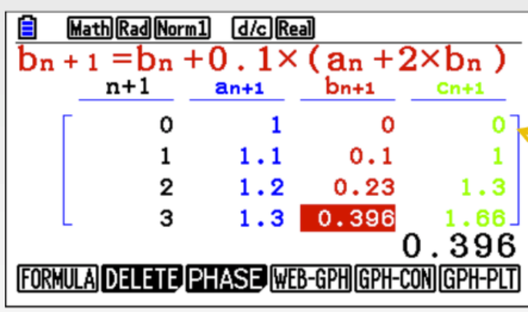
scroll down & highlight y_3





Using the arrows to select each value in the table gives the maximum precision available.

Transfer the results



Ignore the first value in c_{n+1}

x	y	$\frac{dy}{dx}$	$\delta y = h \times \frac{dy}{dx}$
1	0	1	
1.1	0.1	1.3	
1.2	0.23	1.66	
1.3	0.396		

Fill in last column by doing h x your previous value for c_n+1 and running again.

Integrating factor

KEY POINT 5.4

Given a first order linear differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

multiply through by the integrating factor, $I(x) = e^{\int P(x)\,dx}$,

and solve the resulting differential equation.

Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x)\,dx}$

Integrating Factor Method

$$\frac{dy}{dx} + P(x)y = Q(x)$$ * rearrange to this

$$I(x)\frac{dy}{dx} + I(x)P(x)y = I(x)Q(x)$$ * multiply by I(x)

$$\frac{d}{dx}\left(I(x)y\right) = I(x)Q(x)$$ ignore!

$I(x)y = \int I(x)Q(x)$ * write down this

$$y = \frac{1}{I(x)}\int I(x)Q(x)\,dx$$ solve!

Solve $x\frac{dy}{dx} = \cos x - y$.

Step (1) rearrange into $\frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\cos x}{x}$$

$P(x) = \frac{1}{x}$ $\quad Q(x) = \frac{\cos x}{x}$

step (2) Find $I(x)$ [integrating factor]

$$I(x) = e^{\int p(x)}$$

$$I(x) = e^{\int 1/x}$$. Now $\int x^{-1} = \ln x$

$$I(x) = e^{\ln x} = e^{\ln x} = \underline{\underline{x}}$$

step (3)
Multiply equation by $I(x) = x$

$$x\frac{dy}{dx} + \frac{y}{x}x = \frac{\cos x}{x}x$$

step (4) $\quad I(x)y = \int \cdot I(x)Q(x)\,dx$ $\qquad y = \frac{1}{I(x)}\int I(x)Q(x)\,dx$

$$xy = \int \frac{\cos x}{x}x$$

$$y = \frac{1}{x}\int \frac{\cos x}{x}x$$

$$y = \frac{1}{x}\int \cos x$$

$$y = \frac{1}{x}\left[+\sin x + c\right]$$

Solve the differential equation $\cos x \frac{dy}{dx} - 2y\sin x = 3$:

where $y = 1$ when $x = 0$.

step (1) rearrange into $\frac{dy}{dx} + P(x)y = Q(x)$

$$\cos x \frac{dy}{dx} - 2y\sin x = 3$$

$$\Rightarrow \frac{dy}{dx} - 2y\frac{\sin x}{\cos x} = \frac{3}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} - (2\tan x)y = 3\sec x$$

$Q(x) = 3\sec x$

$p(x) = -2\tan x$

step (2) Find $I(x)$ [integrating factor]

$I(x) = e^{\int p(x)}$

$I(x) = e^{-2\int \tan x}$

$* \int \tan x = \int \frac{\sin x}{\cos x}$

$= -\ln(\cos x)$

$I(x) = e^{2\ln(\cos x)}$

$I(x) = e^{\ln(\cos x)^2} = \underline{\underline{\cos^2 x}}$

step (3) multiply by $I(x)$

$$\frac{dy}{dx}-(2\tan x)y=3\sec x$$

$$\cos^2 x\,\frac{dy}{dx} - 2\tan x.\cos^2 x\, y = 3\cos^2 x\sec x$$

step (4) $I(x)y = \int . I(x)Q(x)\,dx$ $\qquad y=\frac{1}{I(x)}\int I(x)Q(x)\,dx$

$$\cos^2 x\, y = \int \cos^2 x.3\sec x$$

$$y = \frac{1}{\cos^2 x}\int \cos^2 x.\,3\sec x$$

$$y = \sec^2 x.3\int \cos x$$

$$y = \sec^2 x.3[\ \sin x + c]$$

step (5) find c. $\qquad y=1 \qquad x=0$

$$y = \sec^2 x.3[\ \sin x + c]$$

$$1 = \sec^2(0).3[\sin 0 + c]$$

$$1 = 3c$$

$$1/3 = c$$

$$y = 3\sec^2 x\left[\sin x + 1/3\right]$$

expect integrals with $\ln f(x)$: especially :

- $\int \tan x = \int \frac{\sin x}{\cos x}$ $\quad * \int \frac{f'(x)}{f(x)} *$

$\int \tan x = -\ln|\cos x|$

$\therefore \quad e^{\int \tan x} = e^{-\ln|\cos x|} = e^{\ln|\cos x|^{-1}}$

$= \frac{1}{\cos x}$

- $\int \frac{1}{x \ln x} = \ln(\ln x)$

$\therefore \quad e^{\int \frac{1}{x\ln x}} = e^{\ln(\ln x)} = \underline{\ln x}$.

Topic 3

Essential topics:
Fundamental Theorem of Calculus
Improper Integrals
Riemann Sums

Fundamental Theorem of Calculus

For a continuous function $f(x)$ on the interval $[a, b]$:

- $$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)$$

any number.

* $$\int_a^b f(t) = F(b) - F(a)$$

* $$\int_x^a f(t)\,dt = -\int_a^x f(t)\,dt$$

e.g.

$$\frac{d}{dx}\int_a^x t^2 + 3\,dt$$

$$\frac{d}{dx}\left[\frac{t^3}{3} + 3t\right]_a^x$$

$$\frac{d}{dx}\left[\frac{x^3}{3} + 3x - a^3 - 3a\right]$$

$$= x^2 + 3$$ [as a is a constant]

Using FTC we get result immediately

$$\frac{d}{dx}\int_a^x t^2 + 3\,dt = \underline{\underline{x^2 + 3}}$$

$F(x) = \int_x^3 (1+t^{16})^{0.5}\,dt$. Find $F'(x)$.

$$F(x) = -\int_3^x (1+t^{16})^{0.5}\,dt$$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(-\int_3^x (1+t^{16})^{0.5}\,dt\right)$$

$$F'(x) = -(1+x^{16})^{0.5}$$

Possible exam style question:

* $5x^3 + 40 = \int_c^x f(t)\,dt$.

Find $f(x)$ and c.

step (1) differentiate both sides :

$$\frac{d}{dx}(5x^3 + 40) = \frac{d}{dx}\int_c^x f(t)\,dt$$

$$15x^2 = \frac{d}{dx}\int_c^x f(t)\,dt$$

$$15x^2 = f(x) \quad \checkmark$$

note $f(t)$

step (2) $5x^3 + 40 = \int_c^x 15t^2\,dt$

$5x^3 + 40 = [5t^3]_c^x$

$5x^3 + 40 = 5x^3 - 5c^3$

$\frac{40}{-5} = c^3$

$-2 = c$

Improper Integrals

Integrals of the form $\int_a^\infty f(x)\,dx$ are known as **improper integrals.**

KEY POINT 2.2

The improper integral $\int_a^\infty f(x)\,dx$ is convergent if the limit

$$\lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}\{I(b)\} - I(a)$$

exists and is finite. Otherwise the integral diverges.

if get ∞

Evaluate $\int_0^{\infty} e^{-3x}\ dx$.

$$\int_0^{\infty} e^{-3x}\ dx = \lim_{b\to\infty}\int_0^{b} e^{-3x}\ dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_0^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}+\frac{1}{3}\right)$$

$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}\right)+\frac{1}{3}$$

$$= \frac{1}{3}$$

$$\int_1^{\infty} \frac{1}{x}\ dx$$

$$= \lim_{b\to\infty}\int_1^{b} 1/x\ dx$$

$$= \lim_{b\to\infty}\left[\ln x\right]_1^b$$

$$= \lim_{b\to\infty}\left[\ln b - \ln 1\right]$$

$$= \underline{\underline{\infty}} \qquad \therefore \int_1^{\infty} \frac{1}{x} \text{ is not convergent (it diverges)}$$

Evaluate the convergent improper integral $\int_1^\infty xe^{-x}\ dx$.

$$\begin{aligned}
\int_1^\infty xe^{-x}\ dx &= \lim_{b\to\infty}\int_1^b xe^{-x}\ dx \\
&= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_1^b - \int_1^b -e^{-x}dx\right) \\
&= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_1^b - \left[e^{-x}\right]_1^b\right) \\
&= \lim_{b\to\infty}\left\{\left(-be^{-b}+e^{-1}\right)-\left(e^{-b}-e^{-1}\right)\right\} \\
&= 2e^{-1} - \lim_{b\to\infty}\left(\frac{1+b}{e^b}\right)
\end{aligned}$$

By l'Hôpital's Rule:

$$\lim_{b\to\infty}\left(\frac{1+b}{e^b}\right) = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

$$\therefore \int_1^\infty xe^{-x}\ dx = 2e^{-1}$$

* Determine for which values of $p\in\mathbb{R}$, $\int_1^\infty x^p\ dx$ is convergent.

$$\int_1^\infty x^p\ dx = \lim_{b\to\infty}\int_1^b x^p\ dx$$

$$= \begin{cases} \lim_{b\to\infty}\left[\dfrac{x^{p+1}}{p+1}\right]_1^b & \text{if } p\neq -1 \\ \lim_{b\to\infty}\left[\ln x\right]_1^b & \text{if } p=-1 \end{cases}$$

$$\begin{cases} \lim_{b\to\infty}\left(\dfrac{b^{p+1}}{p+1}-\dfrac{1^{p+1}}{p+1}\right) & \text{if } p\neq -1 \\ \lim_{b\to\infty}\left(\ln b - \ln 1\right) & \text{if } p=-1 \end{cases}$$

$$\begin{cases} \lim\limits_{b\to\infty}\left(\dfrac{b^{p+1}-1}{p+1}\right) & \text{if } p \neq -1 \\ \lim\limits_{b\to\infty} \ln b & \text{if } p = -1 \end{cases}$$

$$\begin{cases} \infty & \text{if } p > -1 \\ -\dfrac{1}{p+1} & \text{if } p < -1 \\ \infty & \text{if } p = -1 \end{cases}$$

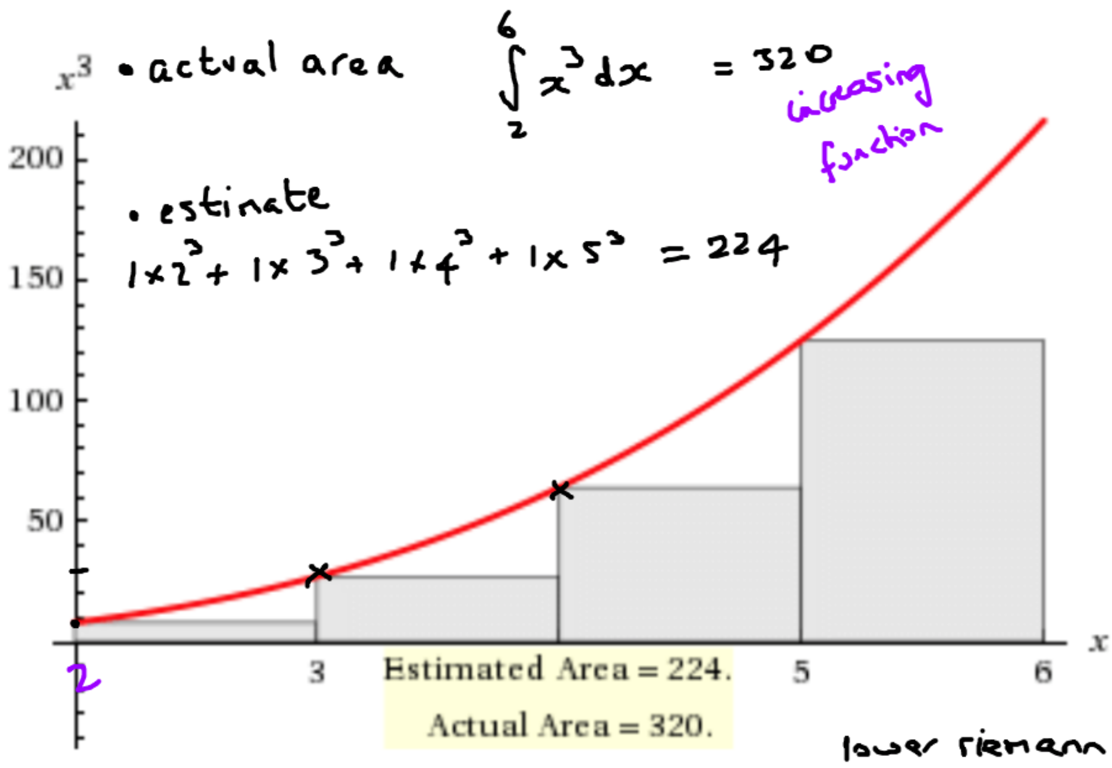
i.e.

$\int_1^\infty x^p \, dx$ converges only for $p < -1$

Or equivalently

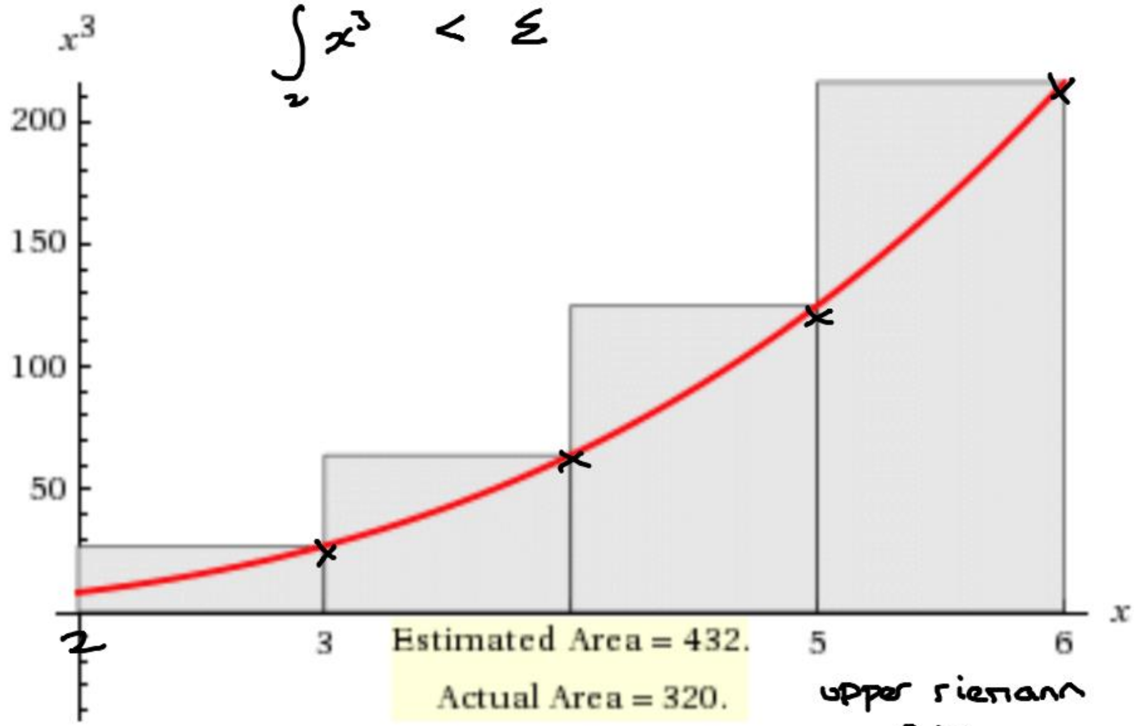
$\int_1^\infty \frac{1}{x^p} \, dx$ converges only for $p > 1$

Riemann Sums



$1 \times 3^3 + 1 \times 4^3 + 1 \times 5^3 + 1 \times 6^3 = 432$

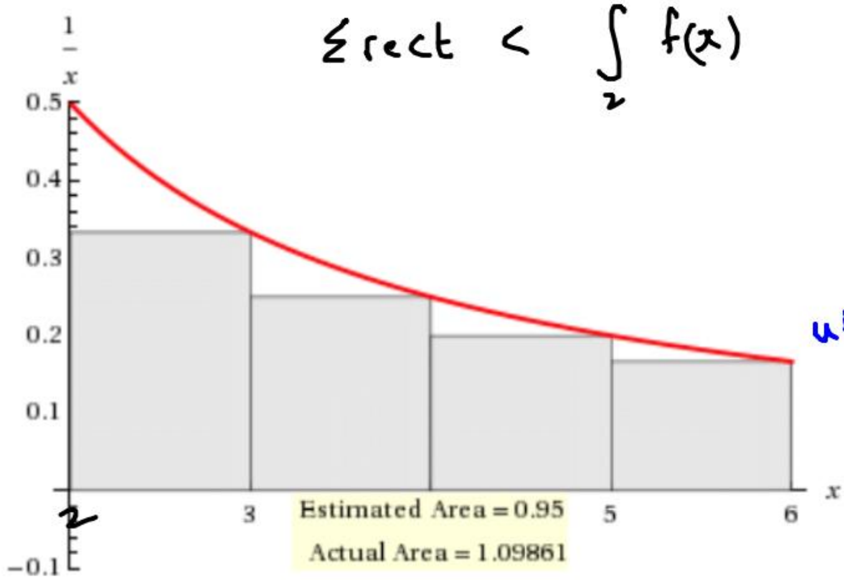
$\int_2^6 x^3 < \Sigma$



upper riemann sum

Note → decreasing function has opposite result!

$\Sigma \text{rect} < \int_2^6 f(x)$



upper Riemann sum

For a decreasing function $f(x)$ for all $x > a$, we have an upper and lower sum such that:

$$\sum_{k=a+1}^{\infty} f(k) < \int_a^{\infty} f(x)\ \mathrm{d}x < \sum_{k=a}^{\infty} f(k)$$

For an increasing function $g(x)$ for all $x > a$, we have an upper and lower sum such that:

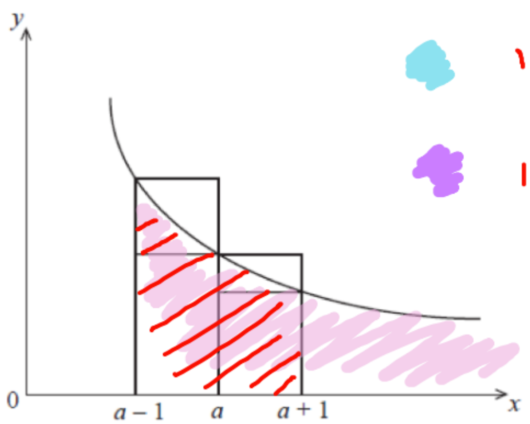
$$\sum_{k=a}^{\infty} g(k) < \int_a^{\infty} g(x)\ \mathrm{d}x < \sum_{k=a+1}^{\infty} g(k)$$

Figure 1 shows part of the graph of $y = \frac{1}{x}$ together with line segments parallel to the coordinate axes.

(i) By considering the areas of appropriate rectangles, show that

$$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}.$$

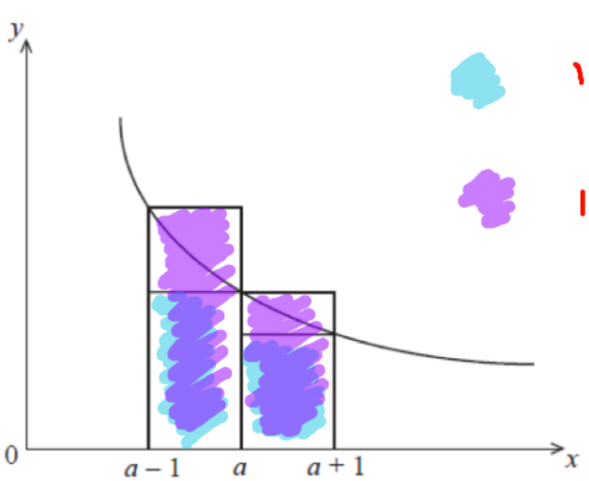
(ii) Hence find lower and upper bounds for $\ln(1.2)$.



$1 \times \frac{1}{a} + 1 \times \frac{1}{a+1}$ Smaller

$1 \times \frac{1}{a-1} + 1 \times \frac{1}{a}$ larger

$\int_{a-1}^{a+1} \frac{1}{x}\,dx$ middle



$1 \times \frac{1}{a} + 1 \times \frac{1}{a+1} + \ldots$

$1 \times \frac{1}{a-1} + 1 \times \frac{1}{a} + \ldots$

$1 \times \frac{1}{a} + 1 \times \frac{1}{a+1}$ smaller LB

$1 \times \frac{1}{a-1} + 1 \times \frac{1}{a}$ larger UB

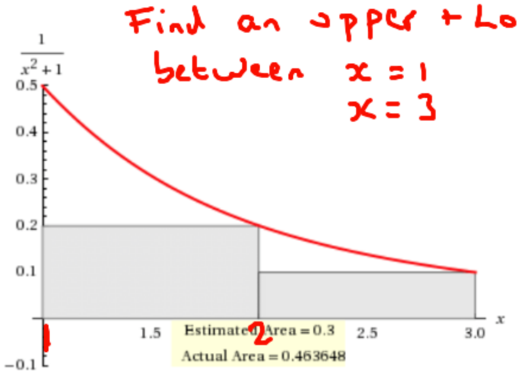
$\int_{a-1}^{a+1} \frac{1}{x}\,dx$ middle

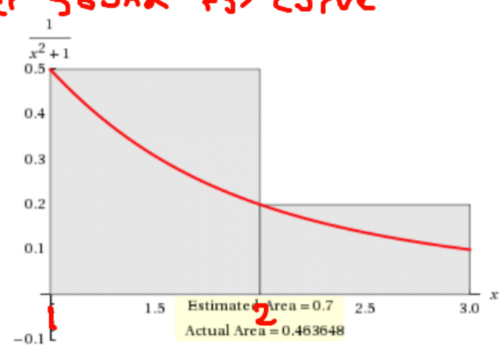
$$\frac{1}{a} + \frac{1}{a+1} < \int_{a-1}^{a+1} \frac{1}{x} < \frac{1}{a-1} + \frac{1}{a}$$

$$\frac{a+1+a}{a(a+1)} < \ln(a+1) - \ln(a-1) < \frac{a+a-1}{(a-1)a}$$

$$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)} \quad \checkmark$$

Find an upper + Lower bound for curve between $x = 1$, $x = 3$

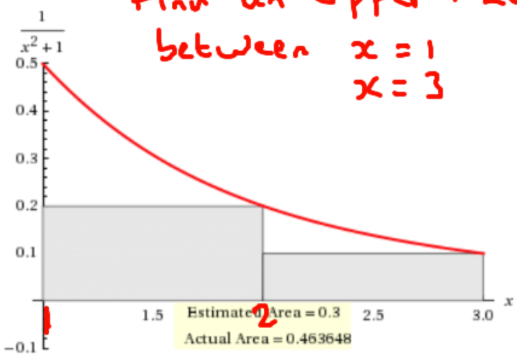


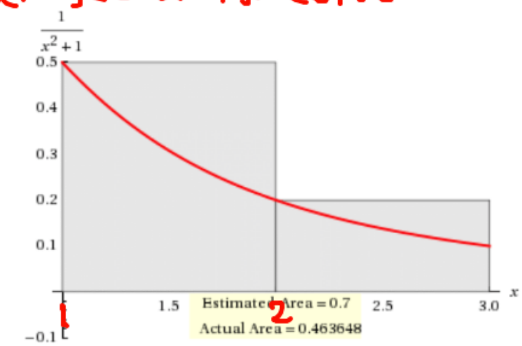


Hence, show that :

$$\frac{3}{10} + \frac{1}{4}\pi < \arctan 3 < \frac{7}{10} + \frac{1}{4}\pi$$

Find an upper + Lower bound for curve between $x = 1$, $x = 3$





$$\frac{1}{2^2+1} + \frac{1}{3^2+1} < \int_1^3 \frac{1}{1+x^2} < \frac{1}{1^2+1} + \frac{1}{2^2+1}$$

$$\frac{1}{5} + \frac{1}{10} < \int_1^3 \frac{1}{1+x^2} < \frac{1}{2} + \frac{1}{5}$$

$$\frac{1}{5} + \frac{1}{10} < \int_1^3 \frac{1}{1+x^2} < \frac{1}{2} + \frac{1}{5}$$

$$\frac{3}{10} < \int_1^3 \frac{1}{1+x^2} < \frac{7}{10}$$

$$\frac{3}{10} < \arctan 3 - \arctan 1 < 7/10$$

$$\frac{3}{10} + \frac{1}{4}\pi < \arctan 3 < \frac{7}{10} + \frac{1}{4}\pi$$

Topic 4

Essential topics:

Convergence of series using;

1) Integral test
2) Divergence test
3) Geometric sum to infinity
4) Comparison test
5) Limit comparison test
6) Ratio test
7) Alternating series test
8) Absolute and conditional convergence

Integral test

Integral Test

Given a positive decreasing function $f(x)$, $x \geq 1$,

if $\int_1^\infty f(x)\,dx$ is:

- convergent then $\sum_{k=1}^{\infty} f(k)$ is convergent
- divergent then $\sum_{k=1}^{\infty} f(k)$ is divergent.

determine if $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges or diverges

$$\int_1^{\infty} \frac{1}{x\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1.5}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-0.5}}{-0.5} \right]_1^b$$

$$= \lim_{b\to\infty} \frac{-2}{\sqrt{b}} + 2$$

$= 2$. so series converges ✓

show that the series

$\sum_{k=1}^{\infty} k^{-1}$ diverges *important result*

$$\sum_{k=1}^{\infty} k^{-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

[called the harmonic series]

you would expect this to converge as

the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$ converges to 0

But no!

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln x\right]_1^b = \underline{\underline{\infty}}$$

$\therefore$ as $\int_1^\infty \frac{1}{x}\,dx$ diverges then $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

geometric series arithmetic series

By finding the nth partial sum of the following series, determine whether the s... diverge.

(a) $\sum_{k=1}^{\infty} 2\times\left(\frac{2}{3}\right)^k$ (b) $\sum_{k=1}^{\infty} \frac{3-k}{4}$

$\frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \ldots$

$S_1 = \frac{4}{3}$

$S_2 = \frac{4}{3} + \frac{8}{9} = \frac{20}{9}$

$S_3 = \frac{4}{3} + \frac{8}{9} + \frac{16}{27} = \frac{76}{27}$

Find S_∞ ....

The sum of n terms of a finite geometric sequence	$S_n = \frac{u_1(r^n-1)}{r-1} = \frac{u_1(1-r^n)}{1-r},\ r \neq 1$
The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1-r},\ \lvert r\rvert < 1$

$$S_n = \frac{\frac{4}{3}\left(\left(\frac{2}{3}\right)^n - 1\right)}{\frac{2}{3} - 1}$$

$$\lim_{n\to\infty} S_n = \frac{\frac{4}{3}(-1)}{-1/3} = 4$$

$$S_\infty = \frac{\frac{4}{3}}{1 - \frac{2}{3}}$$

$$S_\infty = 4$$

(b) $\sum_{k=1}^{\infty} \frac{3-k}{4}$

$\frac{2}{4} + \frac{1}{4} + \frac{0}{4} - \frac{1}{4} \dots$

arithmetic series

$u_1 = \frac{2}{4} \quad d = {}^{-}1/4$

$S_n = \frac{n}{2}\left[2u_1 + (n-1)\,d\right]$

$S_n = \frac{5n - n^2}{8}$

$\lim_{n \to \infty} S_n = \infty$ so diverges.

Divergence test

KEY POINT 3.3

Divergence Test

If $\lim_{k \to \infty} u_k \neq 0$ or if the limit does not exist, the series $\sum_{k=1}^{\infty} u_k$ is divergent.

if $\lim_{k \to \infty} u_k = 0$ then we have no information on whether series converges or not.

Show that the series $\sum_{k=1}^{\infty} \frac{k^2+3k+1}{4k^2+3}$ diverges.

To show that $\lim_{k\to\infty} u_k \neq 0$ we need to manipulate u_k into a form that enables us to find its limit as $k \to \infty$

$$u_k = \frac{k^2+3k+1}{4k^2+3}$$

$$= \frac{1+\frac{3}{k}+\frac{1}{k^2}}{4+\frac{3}{k^2}}$$

$$\therefore \lim_{k\to\infty} u_k = \frac{1}{4} \neq 0$$

Hence $\sum_{k=1}^{\infty} \frac{k^2+3k+1}{4k^2+3}$ diverges.

Comparison test

KEY POINT 3.4

Comparison Test

Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $a_k \leq b_k$ for all $k \in \mathbb{Z}^+$, then if:

- $\sum_{k=1}^{\infty} b_k$ is convergent to a limit S, $\sum_{k=1}^{\infty} a_k$ is also convergent to a limit T where $T \leq S$
- $\sum_{k=1}^{\infty} a_k$ is divergent, so is $\sum_{k=1}^{\infty} b_k$.

useful : $\sum_{k=1}^{\infty} ar^k$ converges if $|r| < 1$

[Sum to infinity geometric]

: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$

[or $\sum_{k=1}^{\infty} k^p$ converges for $p < -1$]

Establish whether or not the series $\sum_{k=1}^{\infty} \frac{1}{2^k+3}$ converges.

geometric/ p series useful

$\sum a_k$: $a_k = \frac{1}{2^k+3}$

$\sum b_k$: $b_k = \frac{1}{2^k}$

and $a_k \leq b_k$

ie $\frac{1}{2^k+3} \leq \frac{1}{2^k}$ ✓

The series is similar to $\sum_{k=1}^{\infty} \frac{1}{2^k}$ which we know converges (it is a geometric series with $r = \frac{1}{2}$), so let's start by considering this

$$\frac{1}{2^k+3} < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{Z}^+$$

and since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, so does $\sum_{k=1}^{\infty} \frac{1}{2^k+3}$ by the Comparison test.

$$\sum \frac{\sin^2 n}{3^n} = \sum a_k$$

- Compare with similar series :

$$\sum \frac{1}{3^n} = \sum b_k$$ This is geometric $r = 1/3$

and

$$\frac{\sin^2 n}{3^n} \leq \frac{1}{3^n}$$ as $\sin^2 \leq 1$

So $a_k \leq b_k$ for all k.

$\sum b_k$ converges $\therefore$ $\sum a_k$ converges.

$$\sum \frac{n}{3n+1} \text{ for } n \geq 4 = \sum b_k.$$

- Compare with $\sum \frac{1}{n} = a_k$ diverges [p series]

$$\frac{1}{n} \leq \frac{n}{3n+1} = \frac{1}{3+1/n}$$ for all $n \geq 4$ ✓

So as $\sum 1/n$ diverges

$\sum \frac{n}{3n+1}$ also diverges.

When does comparison test not work?

eg if $\sum a_k$ $\sum b_k$

$a_k \leq b_k$

But $\sum b_k$ diverges $\rightarrow$ then no information about $\sum a_k$.

OR $\sum a_k$ converges $\rightarrow$ then no information about $\sum b_k$.

in this situation use the limit comparison test

For example the series $\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$ $= \sum b_k$.

Compare with $\sum_{k=1}^{\infty} \frac{1}{2^k}$ $= \sum a_k$ converges geometric $r = 1/2$

$$\frac{1}{2^k} \leq \frac{1}{2^k - 1}$$

But $\sum \frac{1}{2^k}$ converges so no information about $\sum \frac{1}{2^k - 1}$

Limit Comparison

KEY POINT 3.5

Limit Comparison Test

Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $\lim_{k\to\infty} \frac{a_k}{b_k} = l > 0$, then if one series converges so does the other and if one series diverges so does the other.

EXAM HINT

Choose as b_k the general term of the series to which you had hoped to apply the Comparison Test.

Show that the series $\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$ is convergent.

Let

$$a_k = \frac{1}{2^k - 1} \quad \text{and} \quad b_k = \frac{1}{2^k}$$

this is what we hoped to use.

Then

$$\frac{a_k}{b_k} = \frac{1}{2^k - 1} \times \frac{2^k}{1}$$

$$= \frac{2^k}{2^k - 1}$$

$$= \frac{1}{1 - \left(\frac{1}{2}\right)^k}$$

and so

$$\lim_{k\to\infty} \frac{a_k}{b_k} = 1$$

Hence $\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$ converges by the Limit Comparison Test.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} = a_n \qquad \sum \frac{1}{n} = b_n$$

what we wanted to use

Limit Comparison Test

Given two series of positive terms $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $\lim_{k\to\infty} \frac{a_k}{b_k} = l > 0$, then if one series converges so does the other and if one series diverges so does the other.

$$\lim_{n\to\infty} \frac{n}{n^2+1} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1. \quad \therefore \text{ as } \sum \frac{1}{n} \text{ diverges}$$

then $\sum \frac{n}{n^2+1}$ diverges

How to select p series?

Choose same n^{th} term order as original series:

eg $\sum \frac{1}{\sqrt{5n-1}}$ choose $\sum \frac{1}{\sqrt{n}}$

$\sum \frac{2n^2-3}{3n^5+n^3}$ choose $\sum \frac{n^2}{n^5} = \sum \frac{1}{n^3}$

Alternating Series

Alternating series

We have just seen that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges,

but what about the same series with alternating positive and

negative terms?

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

Alternating Series Test

If for an alternating series $\sum_{k=1}^{\infty} u_k$:

- $|u_{k+1}|<|u_k|$ for sufficiently large k
- $\lim_{k\to\infty}|u_k|=0$

then the series is convergent.

V EXAMPLE 1 The alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

satisfies

$$\text{(i)}\quad b_{n+1}<b_n \qquad \text{because} \qquad \frac{1}{n+1}<\frac{1}{n}$$

$$\text{(ii)}\quad \lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{n}=0$$

so the series is convergent by the Alternating Series Test.

Find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ correct to 3 decimal places.

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{24} + \tfrac{1}{120} - \tfrac{1}{720} + \tfrac{1}{5040} + \ldots\ldots$$

Infinite sum:

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} = -\frac{1-e}{e}$: exact value = 0.632120558...

How many terms do we need to add to get an answer accurate to 3dp?

Use truncation error < $|u_{n+1}|$ < 0.0005

so solve $\frac{1}{(n+1)!} < 0.0005$ $\therefore n = 6$

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{24} + \tfrac{1}{120} - \tfrac{1}{720} + \tfrac{1}{5040} + \ldots \doteqdot 0.632 \text{ 3dp}$$

Approximate the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n!}$$

using first 6 terms.

truncation error : Find $u_7 = \frac{1}{7!} = \frac{1}{5040}$

$\therefore$ Sum to infinity approximated by

$$\sum_{1}^{6} (-1)^{n-1} \frac{1}{n!} + \frac{1}{5040}$$

$$0.631944 + \frac{1}{5040} \approx 0.632.$$

How many terms of the series $\sum_{k=1} (-1)^{k-1} \frac{1}{k^2}$
is it necessary to take to find an approximation that is accurate to within 0.001?

$\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\cdots$

$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

truncation error $< |u_{n+1}| < 0.001$

$\frac{1}{(n+1)^2} < 0.001$

$n > 30.62$

So we need 31 terms

ie $\underbrace{1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + \_}_{31 \text{ terms}}$ gets within 0.001

Absolute and Conditional convergence

A series $\sum_{k=1}^{\infty} u_k$ is absolutely convergent if the series $\sum_{k=1}^{\infty} |u_k|$ is convergent.

If a series is absolutely convergent, then it is convergent, i.e.

if $\sum_{k=1}^{\infty} |u_k|$ is convergent then so is $\sum_{k=1}^{\infty} u_k$

If a series $\sum_{k=1}^{\infty} u_k$ is convergent but $\sum_{k=1}^{\infty} |u_k|$ is divergent, then the series is conditionally convergent.

note if $\sum |u_k|$ converges $\Rightarrow$ $\sum u_k$ converges

But $\sum u_k$ converges $\not\Rightarrow$ $\sum |u_k|$ converges !

if a question asks "is the series convergent"
we could use alternating series test
{ test for absolute convergence

But if it asks "is a series convergent and if it is, is it absolutely or conditionally?"

then test both convergence and absolute convergence

if $\sum u_k$ and $\sum |u_k|$ converge then absolute

if $\sum u_k$ converge But $\sum |u_k|$ diverge then conditional

Use of absolute convergence and alternating series test

Show that $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is convergent.

$$|u_k| = \left|\frac{\sin k}{k^2}\right| \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, by comparison test we have $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^2}\right|$ converges

and as absolutely convergent also convergent.

Determine if

$1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \ldots$ Converges.

not alternating series so can't use alternating test.

But $\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

now $|u_n| \leq \frac{1}{n^2}$

so by comparison test $\sum |u_n|$ converges

$\therefore \sum u_n$ converges

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

not alternating so can't use alternating test

If a series is absolutely convergent, then it is convergent, i.e.

$$\text{if } \sum_{k=1}^{\infty} |u_k| \text{ is convergent then so is } \sum_{k=1}^{\infty} u_k$$

$$\sum |u_n| \leq \sum \frac{1}{n^2}$$

now as $\sum \frac{1}{n^2}$ converges, by comparison test we have $\sum |u_n|$ converges absolutely

$\therefore \sum u_n$ is convergent.

$\sum (-1)^n \left(\frac{n^3}{3^n}\right)$

method (1)
alternating:

try alternating series test:

Alternating Series Test

If for an alternating series $\sum_{k=1}^{\infty} u_k$:

- $|u_{k+1}| < |u_k|$ for sufficiently large k
- $\lim_{k\to\infty} |u_k| = 0$

then the series is convergent.

exponentials grow larger than powers

$\frac{(n+1)^3}{3^{n+1}} < \frac{n^3}{3^n}$ ✓ and $\lim_{n\to\infty} \frac{n^3}{3^n} = 0$

$\therefore$ Convergent.

$\sum (-1)^n \left(\frac{n^3}{3^n}\right)$

method (2)

use

If a series is absolutely convergent, then it is convergent, i.e.

if $\sum_{k=1}^{\infty} |u_k|$ is convergent then so is $\sum_{k=1}^{\infty} u_k$

$\sum \frac{n^3}{3^n} \leq \sum \frac{1}{3^n}$ now $\sum \frac{1}{3^n}$ geometric

$\therefore$ converges

So $\sum |u_n|$ is convergent and $\sum u_n$ converges

Absolute or Conditional convergence?

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is the series convergent? If so is it conditionally or absolutely?

step (1) alternating series test:

Alternating Series Test

If for an alternating series $\sum_{k=1}^{\infty} u_k$:

- $|u_{k+1}| < |u_k|$ for sufficiently large k
- $\lim_{k\to\infty} |u_k| = 0$

then the series is convergent.

$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ ✓ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

So convergent.

step (2)

If a series is absolutely convergent, then it is convergent, i.e.

$$\text{if } \sum_{k=1}^{\infty} |u_k| \text{ is convergent then so is } \sum_{k=1}^{\infty} u_k$$

$\sum \frac{1}{\sqrt{n}}$ this is p series with $p = 0.5$

so $\sum |u_n|$ divergent → we therefore have

$\sum u_n$ convergent and $\sum |u_n|$ divergent

∴ conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$ is this absolutely or conditionally convergent?

• alternating series:

$$\frac{1}{\ln(n+2)} \leq \frac{1}{\ln(n+1)} \quad \checkmark$$

$$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0 \quad \checkmark$$

$\therefore$ converges by A.S test

• absolute convergence:

$\sum \frac{1}{n} < \sum \frac{1}{\ln(n+1)}$ so by comparison test, as $\sum \frac{1}{n}$ diverges. $\sum u_n$ is not absolutely so conditionally.

Ratio test

Ratio Test

Given a series $\sum_{k=1}^{\infty} u_k$, if:

- $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| < 1$, then the series is absolutely convergent (and hence convergent)
- $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| > 1$, then the series is divergent
- $\lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| = 1$, then the Ratio Test is inconclusive.

Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge?

$$\lim_{n \to \infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$$

$$\lim_{n \to \infty} \frac{(n+1)}{2^{n+1}} \times \frac{2^n}{n} = \lim_{n \to \infty} \frac{n+1}{2(n)}$$

$$\lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}.$$ Since $L < 1$ converges

$$\sum \frac{2^n}{n!}$$

$$\lim_{n \to \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n}$$

$$\lim_{n \to \infty} \frac{2}{n+1} = 0$$

as $L < 1$ $\therefore$ converges.

EXAMPLE 4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

Summary of tests for convergence

Test	Series	Convergence or Divergence	Comments
nth term divergence test	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	If $\lim_{n\to\infty} a_n = 0$, test is inconclusive
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	Converges to sum, if $S_\infty = \frac{a}{1-r}$, $\lvert r\rvert < 1$. Diverges otherwise.	Useful for comparison tests.
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges when $p > 1$, otherwise diverges.	Useful for comparison tests.
Integral	$\sum_{n=1}^{\infty} a_n;\ a_n = f(n)$	Converges if $\int_1^\infty f(x)dx$ converges; diverges if $\int_1^\infty f(x)dx$ diverges.	$f(x)$ must be continuous, positive, and decreasing.

Comparison	$\sum a_n, \sum b_n;\ a_n \geq 0, b_n \geq 0$	If $\sum b_n$ converges and $a_n \leq b_n$ for all n then $\sum a_n$ converges. If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.	The comparison series is often geometric or a p-series.
Limit comparison	$\sum a_n, \sum b_n;\ a_n \geq 0, b_n \geq 0$	If $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = c,\ c \in R^+$, then both converge or both diverge.	To find b_n consider only terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$	If $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = L$, the series converges (absolutely) if $L < 1$, and diverges otherwise.	Test is inconclusive if $L = 1$.
Alternating	$\sum(-1)^n a_n,\ a_n > 0,$	Converges if $a_k \geq a_{k+1}$ for all k, and $\lim_{n\to\infty} a_n = 0$.	Only applicable to alternating series.
$\sum\|a_n\|$	$\sum a_n$	$\sum\|a_n\| \Rightarrow \sum a_n$ converges.	If $\sum a_n$ converges, but $\sum\|a_n\|$ diverges, then $\sum a_n$ converges conditionally.

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. ~~If a_n is of the form $(b_n)^n$, then the Root Test may be useful.~~

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

V EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

V EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

V **EXAMPLE 5** $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test.

EXAMPLE 6 $\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test.

Essential topics:
Power series – radius of convergence
Taylor and Maclaurin series

A **power series** is an infinite series of the form:

$$\sum_{k=0}^{\infty} a_k(x-b)^k = a_0 + a_1(x-b) + a_2(x-b)^2 + a_3(x-b)^3 + \cdots$$

Often $b=0$ and this reduces to

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

For example a power series representation of

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \ldots$$

sum to infinity of geometric, first term 1, ratio $= -x$

ie $1 - x + x^2 - x^3 = \frac{u_1}{1-r} = \frac{1}{1--x}$ ✓

Power series representation of

$$\frac{1}{1-2x} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots$$

ie S_∞ with $u_1 = 1$ $r = 2x$

$S_\infty = \frac{1}{1-2x}$ ✓

KEY POINT 3.13

The largest number $R \in \mathbb{R}^+$ such that a power series converges for $|x-b| < R$ and diverges for $|x-b| > R$ is called the **radius of convergence** of the power series. It may be determined by the Ratio Test. If:

- $R = \infty$ then the series converges for all $x \in \mathbb{R}$.
- $R = 0$ then the series converges only when $x = b$.

This is nearly a complete description of the range of values for which a power series will converge but since the Ratio Test does not help at the points $x = -R$ and $x = R$ we need to consider these separately each time.

KEY POINT 3.14

The **interval of convergence** of a power series is the set of all points for which the series converges. It always includes all points such that $|x - b| < R$ but may also include end point(s) of this interval.

The exponential series is given by $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the set of values of x for which the series is convergent.

1 If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \div \frac{x^n}{n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1} \cdot n!}{x^n \cdot (n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{n+1} \right|$$

$* \ (n+1)! = (n+1)n! \ *$

$$\lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0$$

$$\therefore \quad \left| \frac{x}{n+1} \right| = 0 < 1 \quad \text{for all } x$$

1 If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Therefore series is convergent for all x

ie $R = \infty$. $[x \in \mathbb{R}]$

Find the interval of convergence of the infinite series

$$\frac{(x+2)}{3\times 1} + \frac{(x+2)^2}{3^2 \times 2} + \frac{(x+2)^3}{3^3 \times 3} + \dots .$$

(10)

$$u_n = \frac{(x+2)^n}{3^n \times n}$$

1 If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\lim_{n \to \infty} \left| \frac{(x+2)^{n+1}}{3^{n+1} \times (n+1)} \div \frac{(x+2)^n}{3^n \times n} \right|$$

$$\lim_{n \to \infty} \left| \frac{(x+2)^{n+1} \cdot 3^n \cdot n}{(x+2)^n \cdot 3^{n+1} \cdot (n+1)} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1} \cdot 3^n \cdot n}{(x+2)^n \cdot 3^{n+1} (n+1)} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x+2)\,n}{3(n+1)} \right| \qquad * \; 3^{n+1} = 3 \cdot 3^n \; *$$

l'hopital with $\underline{\underline{n}}$ $\qquad \frac{\infty}{\infty}$

differentiate w/r to n.

$$\lim_{n\to\infty} \frac{(x+2)}{3} = \left| \frac{x+2}{3} \right|$$

$$\text{limit} = \left| \frac{x+2}{3} \right|$$

1 If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

so series convergent when this limit < 1

$$\left| \frac{x+2}{3} \right| < 1$$

ie $\frac{x+2}{3} < 1 \qquad$ or $\qquad \frac{-(x+2)}{3} < 1$

$x < 1 \qquad\qquad x > -5$

So $-5 < x < 1$ gives convergence

next test boundaries $x = -5$ $x = 1$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

eg: $\left|\frac{1+2}{3}\right| = 1$

When $x = 1$:

$$\frac{(x+2)}{3\times 1} + \frac{(x+2)^2}{3^2\times 2} + \frac{(x+2)^3}{3^3\times 3} + \ldots = \frac{3}{3} + \frac{3^2}{3^2\times 2} + \frac{3^3}{3^3\times 3} + \ldots$$

$$u_n = \frac{3^n}{3^n \times n} = \frac{1}{n}$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent * don't include 1 *

When $x = -5$

$$\frac{(x+2)}{3\times 1} + \frac{(x+2)^2}{3^2\times 2} + \frac{(x+2)^3}{3^3\times 3} + \ldots$$

$$\frac{-3}{3\times 1} + \frac{(-3)^2}{3^2\times 2} + \frac{(-3)^3}{3^3\times 3}$$

$$u_n = \frac{(-1)^n 3^n}{3^n \times n} = \frac{(-1)^n}{n}$$

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - \ldots$ satisfies

$0 \leqslant b_{n+1} \leqslant b_n$ for all $n \in \mathbb{Z}^+$, and if $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

$\lim_{n\to\infty} \frac{1}{n} = 0$

Converges ✓

interval convergence:

$-5 \le x < 1$

V EXAMPLE 1 For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

SOLUTION We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$.

V EXAMPLE 2 For what values of x does the series $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$ converge?

SOLUTION Let $a_n = (x-3)^n/n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\frac{1}{n}}|x-3| \to |x-3| \quad \text{as } n\to\infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\Sigma\, 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\Sigma\,(-1)^n/n$, which converges by the Alternating Series Test. Thus the given power series converges for $2 \leqslant x < 4$.

(a) Consider the power series $\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^k$.

(i) Find the radius of convergence.

(ii) Find the interval of convergence.

1 If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{x}{2}\right)^{k+1}}{k\left(\frac{x}{2}\right)^{k}}\right|$$

$$\lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{x}{2}\right)}{k}\right|$$

$$\lim_{k\to\infty}\frac{(k+1)|x|}{2k}$$

$\frac{\infty}{\infty}$ L'hopital w/r to k.

$$\lim_{k\to\infty} = \frac{|x|}{2}$$

need

$$\frac{|x|}{2} < 1$$

$$|x| < 2 \qquad -2 < x < 2$$

Radius of convergence $= \underline{\underline{2}}$

next check boundary $x = 2$ $x = -2$

at these points:

$$\frac{|x|}{2} = 1$$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

Consider the power series $\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^k$.

when $x = 2$ $\sum_{k=1}^{\infty} k\left(\frac{2}{2}\right)^k = \sum_{k=1}^{\infty} k$ $\underline{\underline{\text{diverge}}}$

when $x = -2$ $\sum_{k=1}^{\infty} k(-1)^k = \sum_{k=1}^{\infty} (-1)^k k$

$$\sum_{k=1}^{\infty} (-1)^k k$$

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \ldots.$ satisfies

$0 \leqslant b_{n+1} \leqslant b_n$ for all $n \in \mathbb{Z}^+$, and if $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

X

Can note $\lim_{k\to\infty} u_k \neq 0$ so diverges

* necessary but not sufficient condition for convergence

So interval of convergence

$$-2 < x < 2$$

Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)3^n}$.

1 If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+2)\,3^{n+1}} \div \frac{(-1)^n x^n}{(n+1)\,3^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1} \cdot (n+1)\,3^n}{(n+2)\,3^{n+1} \cdot x^n (-1)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)\,x\,(n+1)}{3\,(n+2)} \right|$$

$$\lim_{n\to\infty} \frac{|x|(n+1)}{3(n+2)} \qquad \frac{\infty}{\infty} \qquad * \; |-x| = |x| \; *$$

l'hopital w/r to n.

$$\lim_{n\to\infty} = \frac{|x|}{3}$$

$$\frac{|x|}{3} < 1$$

$$|x| < 3$$

$$-3 < x < 3$$

Radius of convergence = 3

Maclaurin and Taylor Series

Maclaurin series	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$
Taylor series	$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$
Taylor approximations (with error term $R_n(x)$)	$f(x) = f(a) + (x-a) f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$
Lagrange form	$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c lies between a and x
Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Find the Maclaurin series for:

(a) $f(x) = e^x$ (b) $g(x) = \sin x$

Give your answers in the form $\sum_{k=0}^{\infty} a_k x^k$

$$\begin{aligned}
&\text{(a)}\quad f(x) = e^x \Rightarrow f(0) = 1 \\
&\quad f'(x) = e^x \Rightarrow f'(0) = 1 \\
&\quad f''(x) = e^x \Rightarrow f''(0) = 1 \\
&\quad f'''(x) = e^x \Rightarrow f'''(0) = 1 \\
&\quad \vdots
\end{aligned}$$

So,

$$\begin{aligned}
f(x) &= \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\
&= \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= \sin x &\Rightarrow\quad & f(0)=0\\
f'(x) &= \cos x &\Rightarrow\quad & f'(0)=1\\
f''(x) &= -\sin x &\Rightarrow\quad & f''(0)=0\\
f'''(x) &= -\cos x &\Rightarrow\quad & f'''(0)=-1\\
f^{(4)}(x) &= \sin x &\Rightarrow\quad & f^{(4)}(0)=0\\
f^{(5)}(x) &= \cos x &\Rightarrow\quad & f^{(5)}(0)=1\\
&\vdots
\end{aligned}$$

So,

$$\begin{aligned}
f(x) &= \frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots\\
&= \frac{0}{0!}+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\frac{0}{4!}x^4+\frac{1}{5!}x^5\cdots\\
&= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}
\end{aligned}$$

Find the Maclaurin series for $f(x)=\ln(1+x)$, giving your answer in the form $\sum a_k x^k$.

$$\begin{aligned}
f(x) &= \ln(1+x) &\Rightarrow\quad & f(0)=0\\
f'(x) &= (1+x)^{-1} &\Rightarrow\quad & f'(0)=1\\
f''(x) &= -(1+x)^{-2} &\Rightarrow\quad & f''(0)=-1\\
f'''(x) &= 2!(1+x)^{-3} &\Rightarrow\quad & f'''(0)=2!\\
f^{(4)}(x) &= -3!(1+x)^{-4} &\Rightarrow\quad & f^{(4)}(0)=-3!\\
f^{(5)}(x) &= 4!(1+x)^{-5} &\Rightarrow\quad & f^{(5)}(0)=4!\\
&\vdots
\end{aligned}$$

So,

$$\begin{aligned}
f(x) &= \frac{f(0)}{0!}+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots\\
&= \frac{0}{0!}+\frac{1}{1!}x-\frac{1}{2!}x^2+\frac{2!}{3!}x^3-\frac{3!}{4!}x^4+\frac{4!}{5!}x^5\cdots\\
&= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}\cdots\\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}
\end{aligned}$$

KEY POINT 4.2

The truncated Maclaurin series:

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

is referred to as the **nth degree Maclaurin polynomial**, $p_n(x)$ of the function $f(x)$.

KEY POINT 4.3

For a function $f(x)$ for which all derivatives evaluated at $x = 0$ exist:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where the error term $R_n(x)$ is given by:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{for some } c \in \,]0, x[$$

This is sometimes referred to as the **Lagrange form of the error term.**

(a) Find an expression for the error term in approximating e^x by its 2nd degree Maclaurin polynomial.

(b) Give an upper bound to 4DP on the error when using this approximation to find $e^{0.75}$.

(a) The 2nd degree Maclaurin polynomial gives the approximation

$$e^x \approx 1 + x + \frac{x^2}{2}$$

with error term

$$R_2(x) = \frac{f^{(3)}(c)x^3}{3!} \qquad c \in \,]0, x[$$

$$= \frac{e^c x^3}{3!}$$

(b) Taking $x = 0.75$ we have

$$R_2(0.75) = \frac{e^c 0.75^3}{3!} \qquad c \in]0, 0.75[$$

$$\therefore R_2(0.75) < \frac{e^{0.75} 0.75^3}{3!} = 0.1489$$

Using the Maclaurin series for $\cos x$, find the series expansion of $\cos(2x^3)$.

We just need to substitute $2x^3$ into the known series for $\cos x$

$$\cos(2x^3) = 1 - \frac{(2x^3)^2}{2!} + \frac{(2x^3)^4}{4!} - \frac{(2x^3)^6}{6!} + \cdots$$
$$= 1 - 2x^6 + \frac{2}{3}x^{12} - \frac{4}{45}x^{18} + \cdots$$

Using the Maclaurin series for $\sin x$ and e^x, find the series expansion of $e^{\sin x}$ as far as the term in x^4.

We start by substituting the series for $\sin x$, only going as far as the x^4 term

$$e^{\sin x} = e^{x - \frac{x^3}{3!} + \cdots}$$
$$\approx e^x e^{-\frac{x^3}{3!}}$$

We now use the series for e^x only going as far as x^4 and then expand

$$\approx \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \left(-\frac{x^3}{3!}\right) + \cdots\right)$$
$$\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^3}{6} - \frac{x^4}{6} + \cdots$$
$$\approx 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

This is known as the Taylor series. All of the results we have used for Maclaurin series generalise in this way.

Taylor approximations

For a function $f(x)$ for which all derivatives evaluated at a exist:

$$f(x)=f(a)+(x-a)f'(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)+R_n(x)$$

where the Lagrange error term $R_n(x)$ is given by

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c\in\,]a,x[$$

(a) Find the Taylor series expansion for $f(x) = \ln x$ around the point $x = 1$.

(b) Using the 4th degree Taylor polynomial as an approximation for this function, find the maximum error for $x \in \left[\frac{1}{2}, \frac{3}{2}\right]$.

(a)

$$\begin{aligned}
f(x) &= \ln x & \Rightarrow\ & f(1) = 0 \\
f'(x) &= x^{-1} & \Rightarrow\ & f'(1) = 1 \\
f''(x) &= -x^{-2} & \Rightarrow\ & f''(1) = -1 \\
f'''(x) &= 2!x^{-3} & \Rightarrow\ & f'''(1) = 2! \\
f^{(4)}(x) &= -3!x^{-4} & \Rightarrow\ & f^{(4)}(1) = -3! \\
f^{(5)}(x) &= 4!x^{-5} & \Rightarrow\ & f^{(5)}(1) = 4! \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{f(1)}{0!} + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots \\
&= \frac{0}{0!} + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 \cdots \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots
\end{aligned}$$

(b) We have:

$$f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + R_4(x)$$

Where,

$$\begin{aligned}
R_4(x) &= \frac{f^{(5)}(c)(x-1)^5}{5!}, \quad c \in \left]\frac{1}{2}, \frac{3}{2}\right[\\
&= \frac{4!c^{-5}(x-1)^5}{5!} \\
&= \frac{(x-1)^5}{5c^5} \\
&< \frac{(x-1)^5}{5\left(\frac{1}{2}\right)^5} \\
&\le \frac{\left(\frac{3}{2}-1\right)^5}{5\left(\frac{1}{2}\right)^5} = \frac{1}{5}
\end{aligned}$$

1. (a) Find the first three terms of the Maclaurin series for $\ln(1 + e^x)$.

(6)

(b) Hence, or otherwise, determine the value of $\lim_{x \to 0} \frac{2\ln(1+e^x) - x - \ln 4}{x^2}$.

(4)
(Total 10 marks)

method (1).

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$\ln(1+e^x)$

$$\ln(1+e^x) = \ln\left(1 + 1 + x + \frac{x^2}{2} + \cdots\right)$$

$$= \ln\left(2\left(1 + \frac{x}{2} + \frac{x^2}{4} + \cdots\right)\right)$$

$$\ln 2 + \ln(1+u) \qquad u = \frac{x}{2} + \frac{x^2}{4} \cdots$$

now use

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\ln 2 + \ln(1+u) = \ln 2 + \frac{x}{2} + \frac{x^2}{4} + \cdots$$

$$-\frac{1}{2}\left(\frac{x}{2} + \frac{x^2}{4}\right)^2 + \cdots$$

$$\ln 2 + \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8}$$

$$\ln 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$$

method (2)

Find the first three terms of the Maclaurin series for $\ln(1 + e^x)$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$f(0) = \ln(1+e^0) = \ln 2$

$f'(x) = \dfrac{e^x}{1+e^x}$ $\qquad$ $f''(x) = \dfrac{e^x(1+e^x) - e^x(e^x)}{(1+e^x)^2}$

$f'(0) = \dfrac{1}{1+1} = \frac{1}{2}$ $\qquad$ $f''(0) = \dfrac{2-1}{4} = \frac{1}{4}$

$\ln(1+e^x) = \ln 2 + x\left(\frac{1}{2}\right) + \frac{x^2}{8}$ ✓

could use L'Hopital

Hence, or otherwise, determine the value of $\lim\limits_{x \to 0} \dfrac{2\ln(1+e^x) - x - \ln 4}{x^2}$.

$2\ln(1+x) = 2\ln 2 + x + \frac{x^2}{4} \ldots +$ higher powers x

$$\lim_{x \to 0} \frac{2\ln 2 + x + \frac{x^2}{4} \ldots + \text{higher powers } x - x - \ln 4}{x^2}$$

$$\lim_{x \to 0} \frac{\frac{x^2}{4} + \text{higher powers } x}{x^2} = \frac{1}{4}$$

(a) Find the Maclaurin series for y up to and including the term in x^2 given that $y = -\frac{\pi}{2}$ when $x = 0$.

(7)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$$f(0) = -\frac{\pi}{2}$$

$$f'(0) = \cos(0) + -\frac{\pi}{2}(\tan(0)) = 1$$

$$f''(x) : \frac{d^2y}{dx^2} - y\sec^2 x - \frac{dy}{dx}\tan x = -\sin x$$

$$\frac{d^2y}{dx^2} = -\sin x + y\sec^2 x + \frac{dy}{dx}\tan x$$

$$\frac{d^2y}{dx^2} = -\sin x + y\sec^2 x + \frac{dy}{dx}\tan x$$

$$f''(0) = -\sin(0) + -\frac{\pi}{2}\sec^2(0) + (1)\tan(0)$$

$$f''(0) = -\frac{\pi}{2}$$

so

$$f(x) = -\frac{\pi}{2} + x(1) + \frac{x^2}{2}\left(-\frac{\pi}{2}\right)$$

$$= -\frac{\pi}{2} + x - \frac{\pi x^2}{4} \ldots$$

3. The function f is defined by

$$f(x) = \ln\left(\frac{1}{1-x}\right).$$

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$.

(1)

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$.

(6)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$$f(0) = \ln\left(\frac{1}{1-0}\right) = 0$$

$f'(x)$: $\ln u \rightarrow \frac{1}{u}$ $\qquad u = (1-x)^{-1}$, $u' = (1-x)^{-2}$

$$f'(x) = \frac{1}{\frac{1}{1-x}}(1-x)^{-2}$$

$$f'(x) = \frac{1-x}{(1-x)^2} = \frac{1}{1-x}$$

$$f''(x) = --(1-x)^{-2} = (1-x)^{-2}$$

$$f'''(x) = --2(1-x)^{-3} \qquad 2(1-x)^{-3}$$

$f'(0) = 1 \qquad f''(0) = 1 \qquad f'''(0) = 2$

$$f(x) = 0 + 1x + \frac{1x^2}{2} + \frac{2x^3}{6}$$ ✓

(c) Use this series to find an approximate value for ln 2.

(3)

(d) Use the Lagrange form of the remainder to find an upper bound for the error in this approximation.

(5)

(e) How good is this upper bound as an estimate for the actual error?

(2)

(Total 17 marks)

$\ln\left(\frac{1}{1-x}\right) \simeq x + \frac{x^2}{2} + \frac{x^3}{3}$

ln 2 want $\frac{1}{1-x} = 2$

$1 = 2 - 2x$

$\frac{1}{2} = x$ so $\ln 2 \simeq \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{2}{3}$

Use the Lagrange form of the remainder to find an upper bound for the error in this approximation.

(5)

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, where c lies between a and x

used $f^3(x)$ so $n = 3$

$R_3(x) = \frac{f^4(c)}{4!}(x-a)^4$

$f^4(x) = 6(1-x)^{-4}$ $\qquad f^4(c) = \frac{6}{(1-c)^4}$

$R_3(x) = \frac{6(x-a)^4}{24(1-c)^4}$

$$R_3(x) = \frac{6(x-a)^4}{24(1-c)^4}$$

our expansion is maclaurin $\therefore$ centred at $x=0$

our approximation for ln 2 used $x = 1/2$

$$R_3(\tfrac{1}{2}) = \frac{6(\frac{1}{2}-0)^4}{24(1-c)^4} = \frac{6(\frac{1}{2})^4}{24(1-c)^4}$$

and have $0 \le c \le 1/2$

$$R_3(\tfrac{1}{2}) \le \frac{6(\frac{1}{2})^4}{24(1-1/2)^4} = \underline{\underline{0.25}}$$

(i) Determine the first three derivatives of the function $f(x) = x(\ln x - 1)$.

(ii) Hence find the first three non-zero terms of the Taylor series for $f(x)$ about $x = 1$.

(7)

(Total 12 marks)

$$f'(x) = \ln x - 1 + 1 = \ln x$$

$$f''(x) = \frac{1}{x}$$

$$f'''(x) = -\frac{1}{x^2}$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

$$f(x) \simeq f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2}f''(1) + \frac{(x-1)^3}{6}f^3(1) + \ldots$$

$$f(x) \simeq f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2}f''(1) + \frac{(x-1)^3}{6}f'''(1)$$

$f(x) = x(\ln x - 1)$. $\quad f(1) = -1$

$f'(x) = \ln x - 1 + 1 = \ln x \quad f'(1) = 0$

$f''(x) = \dfrac{1}{x} \quad f''(1) = 1$

$f'''(x) = -\dfrac{1}{x^2} \quad f'''(1) = -1$

$$f(x) \simeq -1 + \frac{(x-1)^2}{2}(1) + \frac{(x-1)^3}{6}(-1)$$

The End!