HL Mathematics Internal Assessment

Optimization of Packaging Materials

Contents Page

Title Page……………………………..…………………………………………………………..

Contents Page……………………………..…………………………………..……………………1

Important note……………………………..…………………………………..………………...2

Introduction……………………………..……………………………………..…………………...2

Statement of task and rationale…………………………………..…………………………………...2

Cylindrical shaped products…………………………………..…………………………...……..……2-6
- Classic Coca-Cola can…………………………………..…………………………………….…4-5
- Hit Repellent…………………………………..…………………………………...……..……5-6

Cuboidal shaped products…………………………………..……………………………………………6-9
- Nestle Koko-krunch…………………………………..…………………………………..…...7-8
- Mentos Sugar-free Mints…………………………………..…………...…………………......8-9

Conical shaped products…………………………………..……………………………………..……10-14
- Cornetto Oreo…………………………………..…………………………………..………12-13
- Mini Jeli Inaco…………………………………..…………………………………..……...13-14

Conclusion…………………………………..…………………………….……..…………………14-16
- Benefits to society…………………………………..………………………………..……………..…15
- Benefits to businesses…………………………………..……………………………….……..15
- Limitations to society…………………………………..……………………………….………...15
- Limitations to businesses…………………………………..……………………….………………15
- Suggestions for further research…………………………………..……………….…….………..16

Bibliography…………………………………..…………………………………..…………………16
- Internet…………………………………..…………………………………..…………………16
- Books…………………………………..…………………………………..……………...…16

Optimization of Packaging Materials

Important note

This research does not aim to criticize or condemn any particular firms, their choices, or their products. All products are included for academic purposes only.

Introduction

The Sustainable Development Goals (SDGs) were adopted by countries on September 25th, 2015, in order to "end poverty, protect the planet, and ensure prosperity for all"[1]. As we proceed towards sustainable development, we tend to ignore a significant factor that is contributing to societal and environmental deterioration- packaging. I agree that packaging is a necessity- it extends shelf-life of products, helps firms with product differentiation, transports water, and dispenses medications. However, packaging directly relates to "responsible consumption and production", the 12th Sustainable Development Goal, and partially to goals 11, 14 and 17, because it is responsible for "one-third of all municipal trash"[2].

To enhance marketing aspects, firms sacrifice optimisation (or minimization) of packaging materials such as plastic, paper and various metals, which makes packaging a significant contributor to waste. This is a highly threatening case of misallocation of resources.

Statement of task and Rationale

My goal of becoming an impactful Economist propels me towards combining Mathematics and Economics to analyse global issues. Since packaging directly affects the Sustainable Development Goals, I have decided to use the theory of optimisation to calculate the optimal packaging required for common products with cylindrical, cuboidal and conical shapes. First, I have derived formulas for these shapes so that they have the least surface area and utilize minimum packaging materials. Then, using the formula, common products are assessed on how much packaging materials are wasted in their production. The findings are discussed and emailed to manufacturers too, in order to derive pragmatic solutions to reduce packaging waste.

Cylindrical shaped products

First, the size of the cylinder that has the least surface area for a fixed volume V must be calculated because it requires the least materials for packaging.

The volume of a cylinder in terms of radius and height of the cylinder is:

$$V = \pi r^2 h \qquad \text{(Equation 1)}$$ [3]

V = Fixed volume of cylinder in cm^3

r = radius of cylinder in cm

h = height of cylinder in cm

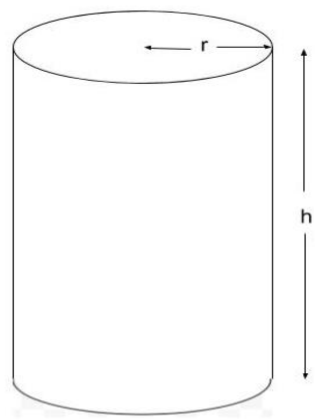


Figure 1: Dimensions of a cylinder

[1] Sustainable development goals. (n.d.). Retrieved September 28, 2017, from http://www.un.org/sustainabledevelopment/sustainable-development-goals/

[2] Reducing Packaging Waste. (2010, December 30). Retrieved September 28, 2017, from http://www.nytimes.com/2010/12/31/opinion/lweb31packaging.html?_r=0

[3] (n.d.). Retrieved October 05, 2017, from http://www.mathopenref.com/cylindervolume.html

When we rearrange Equation 1 in terms of h, we have height in terms of volume and radius:

$$h = \frac{V}{\pi r^2}$$

The equation for the surface area of a cylinder is:

$$S = 2\pi r^2 + 2\pi rh \qquad \text{(Equation 2)}$$ [4]

S = Surface Area in cm^2

This can be written as:

$$S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$S = 2\pi r^2 + \frac{2V}{r}$$

Now we can differentiate surface area with respect to r (V is a constant):

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}$$

At stationary points[5], the surface area reaches a maxima or minima. At these points, the first derivative equals 0. This is applied below:

$$0 = 4\pi r - \frac{2V}{r^2}$$

$$\frac{2V}{r^2} = 4\pi r$$

$$2V = 4\pi r^3$$

$$V = 2\pi r^3$$

The first derivative can be differentiated to determine whether the stationary point is a maxima or minima. Be the second derivative positive, it is a minima. Be it negative, it is a maxima.

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4V}{r^3}$$

Since, $V = 2\pi r^3$,

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4(2\pi r^3)}{r^3}$$

$$\frac{d^2S}{dr^2} = 4\pi + 8\pi = 12\pi$$

12π is positive:

$$12\pi > 0$$

Hence, surface area is minimum when $V = 2\pi r^3$.

Using Equation 1, we can find the relationship between height and radius:

$$\pi r^2 h = 2\pi r^3$$

$$h = 2r$$

Therefore, in a cylindrical shape of constant volume, the surface area is minimum when the **height equals diameter**.

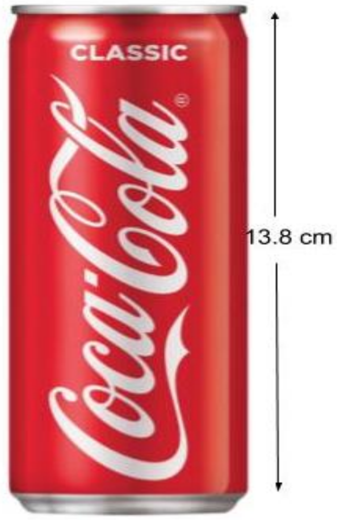
The formulas derived above can be applied to different cylindrically shaped industrial products to determine the optimality of their packaging methods.

[4] (n.d.). Retrieved October 05, 2017, from http://www.aaamath.com/exp79x10.htm

Classic Coca-Cola can

It is estimated that 1.9 billion[6] servings of Coca-Cola are consumed every day. From these, 67873309[7] or approximately 68 million servings are consumed in classic Coca-Cola cans. The classic can is one of the biggest contributors of trash.
The dimensions of a classic can are stated below. A ruler was used to take the measurements.

Figure 2: Dimensions of a classic Coca Cola can



$$h = 13.8 \text{ cm}$$
$$r = 2.70 \text{ cm}$$
$$V \text{ according to print} = 330 \text{ cm}^3$$
$$S = 2\pi r^2 + 2\pi rh$$
$$S = 2\pi(2.70)^2 + 2\pi(2.70)(13.8)$$
$$S = 280 \text{ cm}^2$$

Since surface area is minimum when $V = 2\pi r^3$, we can calculate the optimum radius like this:

$$330 = 2\pi r^3$$
$$r = \left(\frac{330}{2\pi}\right)^{\frac{1}{3}}$$
$$r = 3.74 \text{ cm}$$

Height can be defined as both $\frac{V}{\pi r^2}$ and 2r in a cylinder with optimum or minimum surface area. The value of h should be same using both equations:

$$2(3.74) = \frac{330}{\pi(3.74)^2}$$
$$7.48 \text{ cm} \approx 7.51 \text{ cm}$$

The negligible difference in the value of height is because the r value used was rounded off to 3 significant figures. When the r value stated on the GDC calculator (Casio fx-9860GII SD) is used, h is **7.49 cm** using both equations. So, this value will be used as it is more accurate.

The optimum packaging required is:

$$S = 2\pi r^2 + 2\pi rh$$
$$S = 2\pi(3.74)^2 + 2\pi(3.74)(7.49)$$
$$S = 264 \text{ cm}^2$$

The amount of raw materials wasted in the packaging of one classic Coca Cola can is:

$$S_{\text{Used}} - S_{\text{Optimum}} = \text{Raw Materials wasted}$$
$$280 \text{ cm}^2 - 264 \text{ cm}^2 = 16.0 \text{ cm}^2$$

This seems negligible. However, 67873309[8] cans are consumed per day. When we calculate the amount of raw materials wasted per day, we can see the real picture:

$16.0(67873309) = 1085972944 \text{ cm}^2 = 1.09 \times 10^9 \text{ cm}^2$ (approximately 1.1 billion cm^2)

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[6] How many drinks does The Coca-Cola Company sell worldwide each day? (n.d.). Retrieved October 03, 2017, from http://www.coca-cola.co.uk/faq/how-many-cans-of-coca-cola-are-sold-worldwide-in-a-day
[7] Z. (2009, November 20). Zackshapiro. Retrieved October 03, 2017, from http://zackshapiro.com/post/250962065/the-total-number-of-coca-cola-cans-sold-per
[8] Z. (2009, November 20). Zackshapiro. Retrieved October 03, 2017, from http://zackshapiro.com/post/250962065/the-total-number-of-coca-cola-cans-sold-per

The can's radius should increase and the height should decrease. Ironically, the former classic can better displayed these characteristics as it was wider and shorter.

Coca-Cola has acknowledged the suggestion, and clarified its choices clearly. According to them, "if we were to make our cans shorter and wider it wouldn't be as efficient during the filling or distribution of our drinks. A wider can would mean a case is wider and therefore, less cans would fit on one of our drinks pallet." Furthermore, they believe that "the current 330ml can we use is the best compromise between weight and distribution efficiency". Lastly, they replied, "all of our bottles and cans are 100% recyclable and have been since 2012". This clearly justifies Coca-Cola choices and proves that the current shape is pragmatically optimal.

<u>HIT Repellent</u>

Another cylindrically shaped product, seen in almost every household in Indonesia is the HIT repellent. The sprayer at the top is not included because it does not contain any liquid. Furthermore, it is a separate component added to the can later during the production process. This means that the surface enclosed by the sprayer and the can uses packaging too.

Figure 3: HIT Spray can dimensions

$$h = 26.1 \text{ cm}$$
$$d = 5.80 \text{ cm}$$
$$V \text{ according to print} = 720 \text{ cm}^3$$
$$S = 2\pi r^2 + 2\pi rh$$
$$= 2\pi(2.90)^2 + 2\pi(2.90)(26.1)$$
$$= 528 \text{ cm}^2$$

Now the minimum packaging required can be calculated.
Let's calculate the optimum radius first. Surface area is minimum when $V = 2\pi r^3$.

$$720 = 2\pi r^3$$
$$r = \left(\frac{720}{2\pi}\right)^{\frac{1}{3}}$$
$$r = 4.86 \text{ cm}$$

Since $h = \frac{V}{\pi r^2}$ in a cylinder and $h = 2r$ in a cylinder with minimum surface area for a given volume, the value of h, using both equations, should be same here.

$$2(4.86) = \frac{720}{\pi(4.86)^2}$$
$$9.72 \text{ cm} \approx 9.70 \text{ cm}$$

There is a slight difference because the radius used in the calculation was rounded to 2 decimal places. The height, when using the value shown on the GDC calculator (Casio fx-9860GII SD), is **9.71 cm**. This value is used because it is more accurate.

The optimum packaging required is:

$$S = 2\pi r^2 + 2\pi rh$$
$$S = 2\pi(4.86)^2 + 2\pi(4.86)(9.71)$$
$$S = 445 \text{ cm}^2$$

So, the raw materials wasted on packaging per can of HIT spray is:

$$S_{Used} - S_{Optimum} = \text{Raw Materials wasted}$$
$$528\ cm^2 - 445\ cm^2 = 83.0\ cm^2$$

Hence, 83.0 cm^2 of raw materials are wasted per can of HIT spray packaging.

The firm has not responded to the suggestion emailed. Perhaps, their justifications could include that a wider and shorter can would be harder to grab. Yet, I believe that the can could be compressed as it leaves unnecessary head space to show high quantity. Furthermore, the sprayer at the top should be embedded into the container from the start so that packaging materials used on the enclosed surface can be saved and used elsewhere.

Cuboidal Shaped Products

Let's take a cuboid with a fixed volume V from a rectangular piece of raw material with length l and width w, where squares of side length x have been cut from the corners.

The volume of a cuboid in terms of length l–2x, width w–2x, and height x is:

$$V = \text{length} \times \text{width} \times \text{height} \quad \text{(Equation 3)}^{9}$$

$$V = (l-2x)(w-2x)(x)$$

V = fixed volume of cuboid in cm^3

l = length of rectangular piece in cm

w = width of rectangular piece in cm

x = height of cuboid in cm

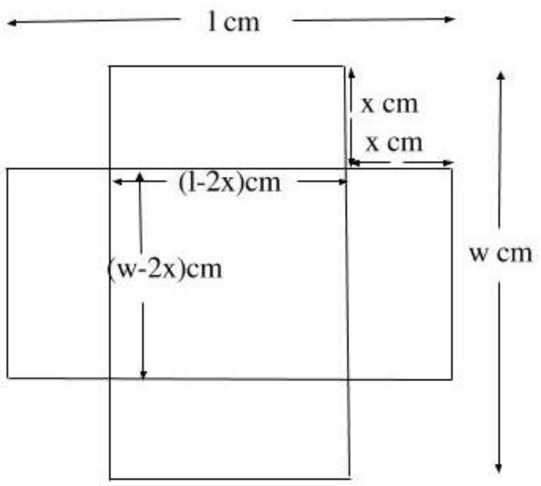

Figure 4: Dimensions of a cuboid

Let's expand the equation for V:

$$V = 4x^3 - 2lx^2 - 2wx^2 + lwx$$

Let's differentiate V with respect to x and keep w and l as constants:

$$\frac{dV}{dr} = 12x^2 - 4lx - 4wx + lw$$

At stationary points, volume will reach a maxima or minima. At these points, first derivative equals 0. This is shown below:

$$0 = 12x^2 - 4lx - 4wx + lw$$
$$0 = 12x^2 - (4)(l + w)x + lw$$

Let's find x using the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a} \qquad \text{(Equation 4)}^{10}$$

[9] (n.d.). Retrieved October 06, 2017, from https://www.mathsisfun.com/cuboid.html
[10] (n.d.). Retrieved October 06, 2017, from https://www.mathsisfun.com/algebra/quadratic-equation.html

$$x = \frac{4(l + w) \pm \sqrt{16(l+w)^2 - 4(12)(lw)}}{2(12)}$$

$$x = \frac{4(l + w) \pm 4\sqrt{(l+w)^2 - 3lw}}{24}$$

$$x = \frac{(l + w) \pm \sqrt{l^2 + w^2 - lw}}{6}$$

For real roots to exist:

$$b^2 - 4(a)(c) \geq 0$$ [11]

$$l^2 + w^2 - lw \geq 0$$

Now, let's go back to the first derivative and differentiate it to determine when V will be maximum.

$$\frac{dV}{dr} = 12x^2 - 4lx - 4wx + lw$$

$$\frac{d^2V}{dr^2} = 24x - 4l - 4w$$

Let's substitute the equation we just derived for x in the second derivative:

$$\frac{d^2V}{dr^2} = 24\left(\frac{(l + w) - \sqrt{l^2 + w^2 - lw}}{6}\right) - 4(l + w)$$

$$= 4((l + w - l - w) - \sqrt{l^2 + w^2 - lw}\,)$$

$$= -4\sqrt{l^2 + w^2 - lw}$$

$$-4\sqrt{l^2 + w^2 - lw} > 0$$

Therefore, when $x = \left(\frac{(l + w) - \sqrt{l^2 + w^2 - lw}}{6}\right)$, V will be maximum.

These calculations have determined a formula for a value of x that leads to the same magnitude of volume V, length l, and width w, but uses optimal packaging materials in cuboidal shaped products. Now, it's time to apply the calculations to common cuboidal products in the market.

<u>Nestle Koko Krunch</u>

Nestle's Koko Krunch cereal brand is taking over the hearts of many children thanks to "the great chocolatey taste"[12] it has. Being my favourite cereal too, I have decided to assess the packaging optimality of the aesthetically pleasing box that it comes in.

Figure 5 Koko Krunch Box

The back surface of the box has been detached because it is not necessary in the investigation.

[11] Quadratic Functions. (n.d.). Retrieved October 06, 2017, from http://www.biology.arizona.edu/biomath/tutorials/quadratic/Roots.html
[12] KOKO KRUNCH®. (2017, April 28). Retrieved October 06, 2017, from https://www.nestle-cereals.com/ph/en/products-promotions/brands/koko-krunch-brand

l = 24.8 cm
w = 32.0 cm
x = 4.90 cm

Figure 6: Dimensions of a Koko Krunch box

The width of the quadrilateral cut from the corners is not 4.9cm like the length. However, it is taken as 4.9 cm to facilitate the investigation.

Let's keep the dimensions except x constant to calculate the packaging materials wasted in a Koko Krunch box

$$x = \left(\frac{(l + w) - \sqrt{l^2 + w^2 - lw}}{6}\right)$$

$$x = \left(\frac{(24.8 + 32.0) - \sqrt{(24.8)^2 + (32.0)^2 - (24.8)(32.0)}}{6}\right)$$

$$x = 4.62 \text{ cm}$$

Since we have assumed the quadrilateral to be a square, this is the formula for the wasted material:

$$x^2_{\text{Used}} - x^2_{\text{Optimum}} = \text{Raw Material Wasted}$$

$$4.90^2 - 4.62^2 = 2.67 \text{ cm}^2$$

Therefore, the amount of raw materials wasted in the packaging of a Koko Krunch box is 2.67 cm^2. This is a relatively low amount indicating that Nestle is operating close to the optimal level of packaging and still managing to keep its Koko Krunch boxes aesthetically pleasing.

The firm has been credited for their operations and also informed of further reductions in packaging waste through an email. My email was acknowledged and even forwarded to the *Legal and Corporate Affairs Director* at Nestle.

Mentos Sugar-free mints

Figure 7 Mentos Sugar-free mints

This is another cuboidal shaped product that has caught the eyes of many. It has a "low calorie density"[13] and is "rich in vitamins and minerals"[14].

The lid of the cuboidal pack has been removed to aid the investigation. The dimensions of the container are shown in figure 8.

[13] Mentos Breath Mints, Sugar Free, Wintergreen. (n.d.). Retrieved October 07, 2017, from https://www.inlivo.com/nutrition/prepared-foods/other-prepared-dishes/mentos-breath-mints-sugar-free-wintergreen
[14] Mentos Breath Mints, Sugar Free, Wintergreen. (n.d.). Retrieved October 07, 2017, from https://www.inlivo.com/nutrition/prepared-foods/other-prepared-dishes/mentos-breath-mints-sugar-free-wintergreen

I first tried to cut the sides in order to lay the container flat like in Figure 4 and Figure 6. However, it was not only dangerous, but would also lead to rough edges which are difficult to measure. Hence, my imagination came into play. Imagine a bird's eye view of the box like in figure 9:

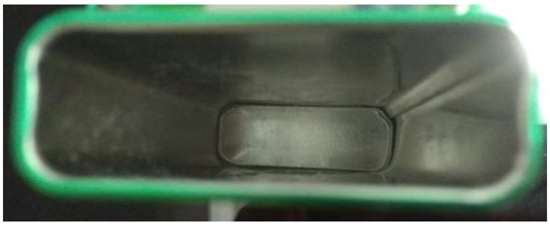

Figure 9: Bird's eye view of container

Now imagine cutting the container so that it is laid flat. A representation is shown in Figure 10.

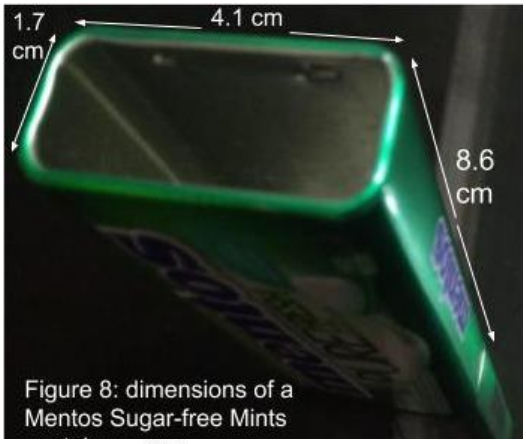

Figure 8: dimensions of a Mentos Sugar-free Mints

Here are the dimensions of the container:

$$l = 21.3 \text{ cm}$$

$$w = 18.9 \text{ cm}$$

$$x = 8.60 \text{ cm}$$

Using these dimensions, we can calculate the packaging materials wasted in Mentos Sugar-free Mints container:

$$x = \left(\frac{(l + w) - \sqrt{l^2 + w^2 - lw}}{6}\right)$$

$$x = \left(\frac{(21.3 + 18.9) - \sqrt{(21.3)^2 + (18.9)^2 - (21.3)(18.9)}}{6}\right)$$

$$x = 3.33 \text{ cm}$$

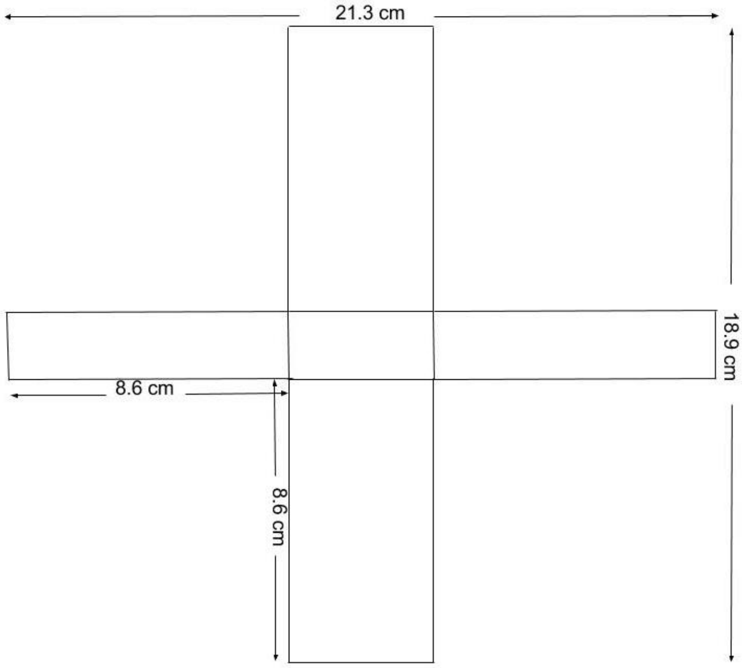

Figure 10: Imagined layout of container

The amount of raw materials wasted are:

$$x^2_{Used} - x^2_{Optimum} = \text{Raw Material Wasted}$$

$$8.60^2 - 3.33^2 = 62.9 \text{ cm}^2$$

So, the amount of raw materials wasted in the packaging of a Mentos Sugar-free Mints container is 62.9 cm^2.

Contrary to the Koko Krunch box, the Mentos Sugar-free Mints container wastes high amounts of materials despite being smaller. The firm "assured this is passed to our manufacturing team", but refrained from providing reasons for its packaging choices. Possible reasons for their choices include customer convenience, aesthetics, and preservation of the can's contents. Yet, the company, financially, and the society, environmentally, is bearing a large expense per container.

The interesting results found in these products, coupled by the companies' responses towards my suggestions have inspired me to delve one step deeper. Let us analyse a more challenging shape- the cone.

Conical Shaped Products

Similar to the cylinder, let's calculate the cone, with a fixed volume V, that has the least surface area as it would require the least materials for packaging.

The volume of a cone in terms of radius and height is:

$$V = \frac{1}{3}\pi r^2 h \text{ (Equation 4)}[15]$$

V = fixed volume of cone in cm^3

r = radius of cylinder in cm

h = height of cylinder in cm

Arranging the equation in terms of h gives us:

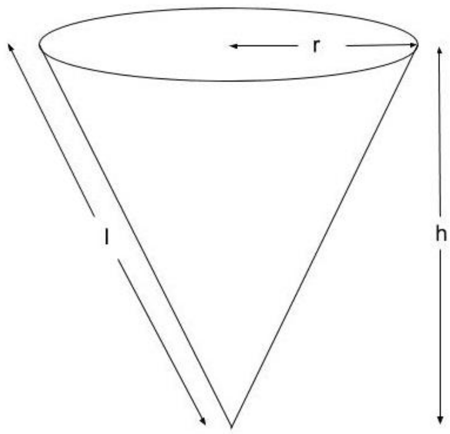

Figure 11: Dimensions of a cone

$$h = \frac{3V}{\pi r^2}$$

The surface area of a cone is:

$$S = \pi rl + \pi r^2 \qquad \text{(Equation 5)}[16]$$

$$l = \sqrt{r^2 + h^2} \qquad \text{(Equation 6)}[17]$$

S = surface area in cm^2

l = slant height in cm

So,

$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$

This equation for surface area can be written in terms of r and V, and further simplified:

$$S = \pi r\sqrt{r^2 + (\frac{3V}{\pi r^2})^2} + \pi r^2$$

$$S = \pi r\sqrt{r^2 + (\frac{9V^2}{\pi^2 r^4})} + \pi r^2$$

$$S = \frac{\pi r\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2} + \pi r^2$$

$$S = \frac{\sqrt{\pi^2 r^6 + 9V^2}}{r} + \pi r^2$$

$$S = \frac{\sqrt{\pi^2 r^6 + 9V^2} + \pi r^3}{r}$$

$$S = \sqrt{\pi^2 r^4 + \frac{9V^2}{r^2}} + \pi r^2$$

This can be written as:

$$S = (\pi^2 r^4 + \frac{9V^2}{r^2})^{\frac{1}{2}} + \pi r^2$$

[15] Formula Volume of a Cone. (n.d.). Retrieved October 07, 2017, from http://www.mathwarehouse.com/solid-geometry/cone/formula-volume-of-cone.php
[16] GCSE Bitesize: Volume and surface area. (n.d.). Retrieved October 07, 2017, from http://www.bbc.co.uk/schools/gcsebitesize/maths/geometry/3dshapeshirev1.shtml
[17] Cone Formula | Area of Cone Formula | Area of Curved Surface Formula. (n.d.). Retrieved October 07, 2017, from http://byjus.com/cone-formula

Now, let's differentiate the equation for S with respect to r (V is a constant):

$$\frac{dS}{dr} = \frac{1}{2}(\pi^2 r^4 + \frac{9V^2}{r^2})^{-\frac{1}{2}}(4\pi^2 r^3 - \frac{18V^2}{r^3}) + 2\pi r$$

$$\frac{dS}{dr} = \frac{4\pi^2 r^4 - \frac{18V^2}{r^2} + 4\pi r\sqrt{\pi^2 r^6 + 9V^2}}{2\sqrt{\pi^2 r^6 + 9V^2}}$$

The surface area will reach a maxima or minima when the first derivative equals 0. This is applied below:

$$0 = \frac{4\pi^2 r^4 - \frac{18V^2}{r^2} + 4\pi r\sqrt{\pi^2 r^6 + 9V^2}}{2\sqrt{\pi^2 r^6 + 9V^2}}$$

We can get rid of the denominator and simplify the equation:

$$0 = 4\pi^2 r^4 - \frac{18V^2}{r^2} + 4\pi r\sqrt{\pi^2 r^6 + 9V^2}$$

$$4\pi r\sqrt{\pi^2 r^6 + 9V^2} = \frac{18V^2}{r^2} - 4\pi^2 r^4$$

$$4\pi r^3\sqrt{\pi^2 r^6 + 9V^2} = 18V^2 - 4\pi^2 r^6$$

$$\pi^2 r^6 + 9V^2 = (\frac{18V^2 - 4\pi^2 r^6}{4\pi r^3})^2$$

$$\pi^2 r^6 + 9V^2 = (\frac{9V^2}{2\pi r^3} - \pi r^3)^2$$

$$\pi^2 r^6 + 9V^2 = \frac{81V^4}{4\pi^2 r^6} + \pi^2 r^6 - \frac{9V^2 \pi r^3}{\pi r^3}$$

$$9V^2 = \frac{81V^4}{4\pi^2 r^6} - 9V^2$$

$$18V^2 = \frac{81V^4}{4\pi^2 r^6}$$

$$4\pi^2 r^6 = \frac{81V^4}{18V^2}$$

$$4\pi^2 r^6 = \frac{9V^2}{2}$$

$$V = (\frac{8\pi^2 r^6}{9})^{\frac{1}{2}}$$

By rearranging, we can derive a formula for r too:

$$4\pi^2 r^6 = \frac{9V^2}{2}$$

$$r^6 = \frac{9V^2}{8\pi^2}$$

$$r = (\frac{9V^2}{8\pi^2})^{\frac{1}{6}}$$

Differentiating the first derivative can determine whether the stationary point is a maxima or minima. If the second derivative is positive, it is a minima. Be in negative, it is a maxima.

$$\frac{dS}{dr} = \frac{4\pi^2 r^4 - \frac{18V^2}{r^2} + 4\pi r\sqrt{\pi^2 r^6 + 9V^2}}{2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$\frac{dS}{dr} = \frac{4\pi^2 r^6 - 18V^2 + 4\pi r^3\sqrt{\pi^2 r^6 + 9V^2}}{2r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

$$\frac{dS}{dr} = \frac{2\pi^2 r^6 - 9V^2}{(\pi^2 r^{10} + 9V^2 r^4)^{\frac{1}{2}}} + 2\pi r$$

$$\frac{d^2S}{dr^2} = \frac{(\pi^2 r^{10} + 9V^2 r^4)^{\frac{1}{2}}(12\pi^2 r^5) - (2\pi^2 r^6 - 9V^2)(\frac{10\pi^2 r^9 + 36V^2 r^3}{2\sqrt{\pi^2 r^{10} + 9V^2 r^4}})}{(\pi^2 r^{10} + 9V^2 r^4)} + 2\pi$$

$$\frac{d^2S}{dr^2} = \frac{(\pi^2r^{10}+9V^2r^4)(24\pi^2r^5)-(2\pi^2r^6-9V^2)(10\pi^2r^9+36V^2r^3)}{2(\pi^2r^{10}+9V^2r^4)^{\frac{3}{2}}} + 2\pi$$

$$\frac{d^2S}{dr^2} = \frac{24\pi^4r^{15}+216V^2\pi^2r^9-20\pi^4r^{15}-72V^2\pi^2r^9+90V^2\pi^2r^9+324V^4r^3)}{2(\pi^2r^{10}+9V^2r^4)^{\frac{3}{2}}} + 2\pi$$

$$\frac{d^2S}{dr^2} = \frac{4\pi^4r^{15}+234V^2\pi^2r^9+324V^4r^3}{2(\pi^2r^{10}+9V^2r^4)^{\frac{3}{2}}} + 2\pi$$; the equation has no negative signs.

$$\frac{4\pi^4r^{15}+234V^2\pi^2r^9+324V^4r^3}{2(\pi^2r^{10}+9V^2r^4)^{\frac{3}{2}}} + 2\pi > 0$$

Since r and V are always positive, $\frac{d^2S}{dr^2}$ is positive for any value of r and V.
This means that surface area is minimum when V = $(\frac{8\pi^2r^6}{9})^{\frac{1}{2}}$.

Using Equation 4, we can find the relationship between height and radius:

$$\frac{1}{3}\pi r^2h = \left(\frac{8\pi^2r^6}{9}\right)^{\frac{1}{2}}$$
$$\frac{1}{9}\pi^2r^4h^2 = \frac{8\pi^2r^6}{9}$$
$$\pi^2r^4h^2 = 8\pi^2r^6$$
$$r^4h^2 = 8r^6$$
$$h^2 = 8r^2$$
$$h = 2r\sqrt{2}$$

Hence, in a conical shape of constant volume, the surface area is minimum when h = $2r\sqrt{2}$.

Using these calculations, we can now evaluate different industrial products, conical in shape, and determine the optimality of their packaging methods.

<u>Cornetto Oreo</u>

Walls[18] is a world-renowned ice-cream brand that sells a popular collection of conical shaped ice-creams known as the Cornetto. From children to adults, everyone likes the Cornetto ice-cream.
This investigation focuses on its newest release that has gained popularity– the Cornetto Disc Oreo[19]. However, all Cornetto ice-creams have the same dimensions approximately, so the findings of this research can be applied to other flavours too.
Using a ruler, I measured the dimensions of **4 Cornetto Oreo ice-creams** and **derived the mean** (because **melting** could have altered the dimensions):

Figure 22 Dimensions of a Cornetto Oreo

h = 15.4 cm
r = 2.8 cm
l = 15.8 cm (not used in calculations)
V according to print = 120 cm^3

[18] Cornetto. (n.d.). Retrieved October 08, 2017, from http://www.walls.com.my/cornetto/
[19] Cornetto. (n.d.). Retrieved October 08, 2017, from http://www.walls.com.my/cornetto/

$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$
$$S = \pi(2.8)\sqrt{(2.8)^2 + (15.4)^2} + \pi(2.8)^2$$
$$S = 162 \text{ cm}^2$$

Using the equation derived for minimum surface area, we can calculate the optimum radius:

$$r = \left(\frac{9V^2}{8\pi^2}\right)^{\frac{1}{6}}$$
$$r = \left(\frac{9(120)^2}{8\pi^2}\right)^{\frac{1}{6}}$$
$$r = 3.43 \text{ cm}$$

Height can be mathematically defined as both $\frac{3V}{\pi r^2}$ and $2r\sqrt{2}$ in a cone with optimal surface area. Hence, the value of h should be same using both equations:

$$\frac{3V}{\pi r^2} = 2r\sqrt{2}$$
$$\frac{3(120)}{\pi(3.43)^2} = 2(3.43)\sqrt{2}$$
$$9.74 \text{ cm} \approx 9.70 \text{ cm}$$

The difference in the calculated values can be neglected as r is an estimate to 3 significant figures. Using the r value in the GDC calculator (Casio fx-9860GII SD), h is **9.71 cm**. So, this value is used to calculate the optimal packaging:

$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$
$$S = \pi(3.43)\sqrt{(3.43)^2 + (9.71)^2} + \pi(3.43)^2$$
$$S = 148 \text{ cm}^2$$

The amount of packaging materials wasted in one cone of Cornetto Oreo is:

$$S_{Used} – S_{Optimum} = \text{Raw Materials wasted}$$
$$162 \text{ cm}^2 – 148 \text{ cm}^2 = 14.0 \text{ cm}^2$$

This seems small. However, Cornetto has been selling worldwide since 1976[20] and, similar to the classic Coca-Cola can, is mass-produced. A small reduction in the ice-cream's surface area can save significant amount of packaging material. Walls, the parent company, is yet to respond to the suggestion. But, speaking from experience, the long, conic shape is the ice-cream's Unique Selling Point (USP).

<u>Mini Jeli Inaco</u>

Inaco Jelly is a popular jelly brand in Indonesia. A pack of 65 jellies costs just 25000[21] rupiah or 1.85 USD, making it an affordable snack for almost everyone. But, we have already seen that with great quantity comes great packaging waste. Therefore, though small, this is an important product to assess.

The slant height has been removed as it was difficult to measure accurately. Furthermore, it is not required.

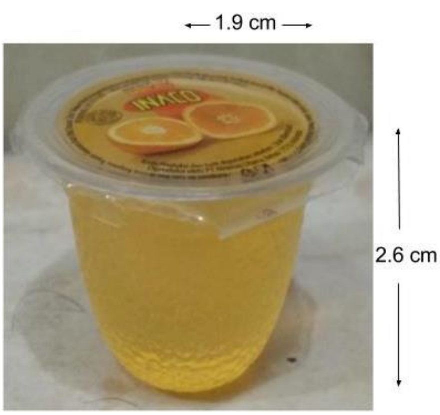

Figure 13 Dimensions of Inaco Jeli

[20] Cornetto. (n.d.). Retrieved October 08, 2017, from https://www.unilever.co.uk/brands/our-brands/cornetto.html
[21] Jual Inaco Jelly Online - Harga & Kualitas Terjamin. (n.d.). Retrieved October 08, 2017, from https://www.blibli.com/inaco-jelly-MTA.0531135.htm

The dimensions were measured using a ruler:

$$h = 2.6 \text{ cm}$$
$$r = 1.9 \text{ cm}$$
$$V \text{ according to print} = 10 \text{ cm}^3$$
$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$
$$S = \pi(1.9)\sqrt{(1.9)^2 + (2.6)^2} + \pi(1.9)^2$$
$$S = 30.6 \text{ cm}^2$$

The optimum radius is:

$$r = \left(\frac{9V^2}{8\pi^2}\right)^{\frac{1}{6}}$$
$$r = \left(\frac{9(10)^2}{8\pi^2}\right)^{\frac{1}{6}}$$
$$r = 1.50 \text{ cm}$$

In this case, height is both $\frac{3V}{\pi r^2}$ and $2r\sqrt{2}$. Both equations should give the same values for h:

$$\frac{3(10)}{\pi(1.50)^2} = 2(1.50)\sqrt{2}$$
$$4.24 \text{ cm} \approx 4.24 \text{ cm}$$

The ≈ sign indicates that although the 3 significant figures values of h are equivalent, the values displayed on the calculator were not exactly the same.

The displayed value of r on the GDC calculator (Casio fx-9860GII SD) also results in **4.24 cm** as the height.

The optimal area and packaging can now be calculated:

$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$
$$S = \pi(1.50)\sqrt{(1.50)^2 + (4.24)^2} + \pi(1.50)^2$$
$$S = 28.3 \text{ cm}^2$$

The amount of packaging materials wasted per Inaco jelly is:

$$S_{Used} - S_{Optimum} = \text{Raw Materials wasted}$$
$$30.6 \text{ cm}^2 - 28.3 \text{ cm}^2 = 2.30 \text{ cm}^2$$

One pack contains 65 jellies. Packaging materials wasted per pack are:

$$2.30 \times 65 = 150 \text{ cm}^2$$

Again, small changes in the packaging design can catalyze huge reductions in packaging waste and help the planet advance one step closer to accomplishing the Sustainable Development Goals. However, the shape of the container does not perfectly abide the shape of a cone, so the calculations are estimates.

This product is sold by a local firm and I was unable to get access to the company’s enquiry contact, annual reports, interviews and customer reviews regarding packaging. However, I believe that the firm focuses on attracting children rather than reducing waste. The curve structure of the jelly has a smooth shape that a child likes.

Conclusion

This investigation has broadened my understandings regarding the business and societal considerations involved in the packaging of a product. Calculus remains an extremely useful tool even outside Kinematics and Marginal Costs.

While the calculations were accurate, some businesses did not respond to my suggestions. Hence, I was unable to formulate concrete pragmatic solutions for all products discussed above. Nonetheless, possible benefits and limitations of packaging reduction are discussed below:

Benefits to society

1. Lots of packaging materials cannot be reused and recycled, for example- cereal box plastic, potato chip bags and soiled tin cans[22]. So, it is best to prevent wastage.
2. "Energy recovery – Waste-to-energy and refuse-derived fuel in approved facilities are able to make use of the heat available from the packaging components."[23]
3. Energy consumption to handle packaging waste is reduced.
4. Lots of waste materials are burned. Reduction in waste leads to reduction in pollution.
5. Waste prevention is crucial for economic development and the Sustainable Development Goals.

Benefits to businesses

1. Improved corporate image by complying to legislative procedures.
2. Reduced packaging costs. For example- cost of materials used in a Mentos Mint Can will be largely reduced.
3. Hiring employees and machinery to manage waste is not necessary.
4. Supply chain is streamlined and focus can be shifted towards innovation and brand image.
5. Stakeholders are satisfied.
6. Eco-friendly firms are actively supported by the government through subsidies and other grants.

Limitations to society

1. The Real GDP or Economic Growth is reduced in the short-run as firms do not produce at full productive capacity.

Limitations to businesses

1. Aesthetic aspect of the product portfolio is somewhat sacrificed.
2. Brand visibility on the product might need to be somewhat sacrificed.
3. Increased risk of theft by retailers due to absence of vivid packaging.
4. Some businesses such as Coca-Cola have already adopted recyclable practices.
5. High amounts of inefficiency due to disruptions in the supply chain as business cannot operate at its desired level.
6. The transmission of information is sacrificed due to reduced surface area.
7. Durability and safety of especially fragile products is sacrificed.

Making profits at the expense of the environment is detrimental to society. Our society must transform into an eco-friendly platform where Corporate Social Responsibilities (CSRs) surround firms. Various steps have already been taken by firms to catalyze sustainable development. For example- Tata Motors, an Indian automobile firm, plants thousands of trees yearly; Coca-Cola production materials are 100% recyclable.

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[22] *Non-recyclable Material List*[PDF]. (n.d.). http://lausd-oehs.org/docs/Recycling/Non_Recyclable_List.pdf

[23] B. (n.d.). Packaging. Retrieved October 16, 2017, from https://courses.lumenlearning.com/boundless-marketing/chapter/packaging/

Suggestions for further research

To further the scope of this investigation, multi-shaped products such as plastic water-bottles can be analyzed. The bottom-half of a bottle can be taken as a cylinder and the upper-half as a cone. This technique can help analyze numerous multi-shaped products that are contributing to packaging waste.

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