Topic 6 Circular motion and gravitation

6.1 Circular motion

1) Essential idea - A force applied perpendicular to a body's displacement can result in its circular motion.

2) The force must act inwards in all cases of circular motion.

3) Data booklet

1. \( v = \omega r \)
2. \( a = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2} \)
3. \( F = \frac{mv^2}{r} = ma^2 r \)

4) Utilization - Motion of charged particles in magnetic fields Unit 5.4

5) A particle is said to be in uniform circular motion if it travels in a circle (or arc) with constant speed \( v \).

Magnitude of velocity does not change.
Direction of velocity does change.
Thus, there is an acceleration although speed does not change.

\[ a = \frac{\Delta v}{\Delta t}, \quad \Delta v = v_2 - v_1 \]

Acceleration is towards the center.
Centripetal = center seeking

\[ F_c = m a_c \text{ [centripetal force]} \]

\( \downarrow \) radius = \( \uparrow \) \( F_c \)
\( \uparrow \) velocity = \( \uparrow \) \( F_c \)
7) \[ a_c = \frac{v^2}{r} \]

- Centripetal acceleration

E.g. 730 kg car negotiates 36 m radius turn at 25 m/s. Centripetal acceleration and force? What force is causing this acceleration?

\[ a_c = \frac{v^2}{r} = \frac{25^2}{30} = \frac{625}{30} = 20.8 \text{ m/s}^2 \]

\[ F_c = ma_c \]

\[ F_c = 730 \times 20.8 = 15208 \text{ N} \]

\[ = 1.5 \times 10^4 \text{ N} \]

The centripetal force is caused by the friction between the tires and the pavement.

8) Period T - Period T is the time for one complete revolution.

Frequency f - Cycles or oscillation or revolutions per second, Hz

\[ \frac{1}{T} \] Period and Frequency

Find period & frequency of one day.

Period T = 24 \times 60 \times 60 = 86400s

Frequency \( f = \frac{1}{86400} = 1.16 \times 10^{-5} \text{ Hz} \)

9) Velocity & period

One revolution is circumference \( C = 2\pi r \)

Therefore \( v = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T} \)

\[ v = \frac{2\pi r}{T} \]

\[ v^2 = \frac{4\pi^2 r^2}{T^2} \]

\[ a_c = \frac{v^2}{r} = \frac{4\pi^2 r^2}{T^2} = \frac{4\pi^2 r}{T^2} \]
Centripetal acceleration

Example

A 3.0 kg cat is swung by string having radius 3.00 meters. He takes 5.00 seconds to complete one revolution. What are $a_c$ and $F_c$?

$$a_c = \frac{v^2}{r}$$

$$F_c = ma_c$$

$$F_c = 2.50 \times 4.74$$

$$F_c = 11.8 \text{ N}$$

$$a_c = \frac{(2\pi)^2}{r}$$

$$a_c = \frac{4\pi^2 (3)^2}{5^2 \times 3}$$

$$a_c = \frac{36 \pi^2}{75}$$

$$a_c = 4.74 \text{ m/s}^2$$

Angular displacement and arc length

Displacement or arc length or distance

$$s = r\theta$$

$\theta$ in radians

Relation between $s$ and $\theta$

Angular displacement

$$\omega = \frac{\theta}{t}$$

Angular velocity

$$\text{rads}^2$$

Angular speed

$$v = \frac{s}{t} = \frac{r\theta}{t} = r \left(\frac{\theta}{t}\right) = r\omega = v$$

Radian-degree-revolution conversions

$\pi \text{ rad} = 180^\circ = \frac{1}{2} \text{ rev.}$

$2\pi \text{ rad} = 360^\circ = 1 \text{ rev}$
**Example**

A mass moving at a constant speed \( v \) in a circle of radius \( r \) as shown.

Find:

a) the period \( T \) of the point mass, and
b) the frequency \( f \) of the point mass, and
c) the angular speed \( \omega \) of the point mass.

\[
\begin{align*}
\text{a) } & T = \frac{2\pi r}{v} \\
\text{c) } & \omega = \frac{v}{r} \times \frac{2\pi}{1}
\end{align*}
\]

\( T = 12 \text{ seconds} \)

(6) Find the angular speed of the second hand on a clock. Find speed of the tip of the hand if it is 18.0 cm long.

\[
\omega = \frac{\theta}{t} = \frac{2\pi}{60} = \frac{\pi}{30} = \frac{\pi}{180} \text{ radians per second}
\]

\[
\omega = \frac{2\pi}{180} \text{ radians per second} = \frac{\pi}{90} \text{ radians per second}
\]

\[
\begin{align*}
\text{speed of the tip of the hand} & = v = kr \omega \\\n& = 18 \times 10^{-2} \times 0.105 \\
& = 0.0188 \text{ m/s}
\end{align*}
\]

(8) Car rounds a 90° turn in 6.0 seconds. Angular speed?

\[
\omega = \frac{\theta}{t} = \frac{\pi}{12} = \frac{0.262 \text{ radians}}{1}\]

**This is banking.** (Considered qualitatively only in syllabus)

\[ \text{As a plane banks (rolls), lift vector begins to have a horizontal component.} \]

\[ \text{The centripetal force causes the plane to begin traveling in a horizontal circle.} \]
Even though cars use friction, roads are banked so that the need for friction is reduced.

A component of the normal force provides centripetal force.

(i.e., an uplifted curve or corner) (Maut ka kula)

(It can turn even if it is frictionless.)

1) Angular speed and centripetal acceleration

\[ v = r\omega \]
\[ a_c = \frac{v^2}{r} = r\omega^2 \]

| \( a_c = \frac{r\omega^2}{T^2} \) | \( F_c = \frac{4\pi^2r}{T^2} \) | \( a_c \) and \( F_c \) (all three forms)
| \( a_c = \frac{4\pi^2r}{T^2} \) | \( F_c = \frac{4\pi^2mr}{T^2} \) |
| \( a_c = \frac{v^2}{r} \) | \( F_c = \frac{mv^2}{r} \) |

\[ \omega = \frac{2\pi}{T} = \frac{2\pi}{t} \] relation between \( \omega \), \( T \) and \( t \)

Angular speed with a direction is called angular velocity.

Direction of a rotation: Right hand rule:
1. Rest heel of your right hand on the rotating object.
2. Make sure your fingers are curled in the direction of rotation.
3. Your extended thumb points in the direction of the angular velocity.

Angular velocity always points perpendicular to the plane of motion.
Example
Find angular velocity (in rad/s) of the wheel on the shaft. It is rotating at 30.0 rpm (revolutions per minute).

\[ \omega = \frac{\theta}{t} = \frac{30 \times 2\pi}{60} = \frac{60 \pi}{60} = \pi \text{ rad/s} \]

Magnitude = 3.14 rad/s
Direction = RHR

(2) Forces providing centripetal forces

Example
Friction (race car making a turn)
Tension (Albert the physics cat spinned)
Gravitational force (baseball and the Earth)
Electric force (an electron orbiting a nucleus)
Magnetic force (a moving charge in a B-field)

Example
1) Dobson watching 16-pound bowling ball being swung around at 50 m/s by Arnold. If string is cut at the instant the ball is next to the ice cream, what will the ball do?

It will fly tangent to the original circular path along c.

2) 0.500 kg baseball is placed in a circular orbit around earth at slightly higher than tallest point 8850 m, mount everest. Given \( R_e = 6400000 \text{ m} \) is radius of Earth. Find the speed of ball.

\[ F_e = \frac{GMm}{r^2} \text{ is caused by gravitational force.} \]

\[ mg = 0.500 \times 9.81 = 4.905 N \]

\[ \frac{4.905 \times 8850}{0.5} = V \]

\[ v = 8000 \text{ m/s} \]
b) How long will it take the ball to return to Everest?

\[ v = \frac{ds}{t} \]

\[ 8000 = \frac{2\pi (640000000)}{T} \]

\[ T = 5033.5 \text{ seconds} = 1.410 \text{ hours} \]

d) Explain how an object can remain in orbit yet always be falling.

When the ball is thrown at a small speed, the ball is drawn by gravity towards the center of the Earth. But, when the ball is finally thrown at a great enough speed that it matches the circumference of the Earth, the ball is effectively falling around the Earth while in orbit.

Q) Angular speed of the minute hand of a clock?

\[ \omega = \frac{\theta}{t} = \frac{2\pi}{3600} \]

\[ = 0.00175 \text{ rad s}^{-1} \]

Angular speed of rotation of the Earth in one day?

\[ \omega = \frac{2\pi}{T} = \frac{2\pi}{60 \times 60 \times 24} = 7.27 \times 10^{-5} \text{ rad s}^{-1} \]

The small angular speed is why we can't feel the Earth as it spins.

Example

Find the apparent weight of someone standing on an equatorial scale if his weight is 882 N at the North pole. 90° N

\[ \omega = 0.0000727 \text{ rad s}^{-1} \]

Lines are line of a latitude.

At the equator, \( r = R \), and

\[ a_c = \sqrt{\omega^2 + \frac{v^2}{R}} = 6.4 \times 10^6 \times 0.0000727 + 9.8988 \text{ m/s}^2 \]

At the pole,

\[ a_c = \omega^2 \]

\[ = 0.0000727^2 = 0 \text{ m/s}^2 \]

\[ \Sigma F = ma \]

\[ W - R = F_c \]

\[ W = F_c + R \]

\[ R = W - ma = 882 - (882 \times 0.00727) = 819.7 \text{ N} \]

Man lost 3 N.
Example

A linear spring of negligible mass requires 18.0 N to cause its length to increase by 1.0 cm. A sphere of mass 75.0 g is attached to one of the spring. Distance between centre of sphere M & other end P of the un-stretched spring is 25.0 cm, as shown below.

Sphere is rotated at constant speed in horizontal circle with centre P. PM increases to 26.5 cm.

a) Explain why the spring increases in length when the sphere is moving in a circle.

\[ F = kx = mv^2, \]  
As velocity increases, force increases to keep it moving in a circle. In this case, \( m \) and \( v^2 \) are proportionally directly proportional to \( F \).

So, \( x \) increases.

b) Determine the speed of the sphere.

\[ F = 18 \times 0.015 = 0.27 \text{ N} \]
\[ 0.27 = 0.075 \times v^2 \]
\[ v = \sqrt{\frac{0.27}{0.075}} = 4.5 \text{ m/s} \]

Solution

18.0 N cm\(^{-1}\) = 1800 N m\(^{-1}\), \( k = 1800 \text{ N m}^{-1}\)

\[ \frac{1800}{2} E = 1800 \times 0.015 = 27 \text{ N} \]

\[ 27 = 0.075 \times v^2 \]
\[ v = \sqrt{\frac{27}{0.075}} = 9.77 \text{ m/s} \]

Example

Point P has linear speed \( v \) and centripetal acceleration \( a \). What about point Q?

\[ v_1 = Rw \]
\[ v_2 = 2Rw \]
\[ 2 \times v_1 = v_2 \]

\[ a_1 = Rw^2 \]
\[ a_2 = 2Rw^2 \]

Centripetal acceleration is always towards center. (center-seeking)

Rotating at constant speed. Draw the horizontal force or forces acting on the brick?

Linear speed: \( 2v \)

Cent. Acceleration \( = 2a \)
6.2 Newton's Law of Gravitation

- Newton's Law of Gravitation
  - The gravitation force is the weakest of all the 4 fundamental forces.

- Newton's law of gravitation - Gravitational force between two point masses \( m_1 \) and \( m_2 \) is proportional to their product, inversely proportional to the square of their separation \( r \).

\[
F = \frac{G m_1 m_2}{r^2}
\]

where \( G = 6.67 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \)

- Universal law of gravitation
- \( G \) is universal gravitational constant.
- Discovered by Henry Cavendish

The celestial bodies we observe are point masses. They have layers.

"A spherically symmetric shell of mass \( M \) acts as if all of its mass is located at its center." - Newton

2) Solving problems involving gravitational force

**Earth**

Assuming each shell is symmetric, the gravitational force caused by that shell acts as though its mass is all concentrated at its center.

Net force at \( m \):

\[
F = \frac{GM_1 m}{r^2} + \frac{GM_0 m}{r^2} + \frac{GM_m m}{r^2} + \frac{GM_c m}{r^2}
\]

\[
F = \frac{G(M_1 + M_0 + M_m + M_c) m}{r^2}
\]

Thus, \( F = \frac{G M m}{r^2} \) where \( M = M_1 + M_0 + M_m + M_c \)

\( M \) is just total mass of Earth lb
Example

Earth has a mass of $M = 5.98 \times 10^{24}$ kg and moon has mass of $m = 7.36 \times 10^{22}$ kg. Mean distance between Earth & moon is $3.82 \times 10^8$ m.
What is the gravitational force between them?

$$F = \frac{GMm}{r^2} = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times 7.36 \times 10^{22}}{(3.82 \times 10^8)^2}$$

$$= \frac{1.68 \times 10^{20}}{2.01 \times 10^{20}} N$$

Radius of each mass is immaterial here because we are already referring to point masses. So, don't worry about adding it here.

Example

Moon mass $7.36 \times 10^{22}$ kg. Mean distance between earth & moon is $3.82 \times 10^8$ m. What is the speed of moon in its orbit about Earth?

$$F_a = 2.01 \times 10^{20} N$$

$$F_g = \frac{F_c}{2.01 \times 10^{20}} = \frac{7.36 \times 10^{22} \times v^2}{3.82 \times 10^8}$$

$$2.01 \times 10^{20} \times 3.82 \times 10^8 = v^2$$

$$v = \sqrt{102.14 \text{ m/s}^2}$$

For circular orbits, the $F_a$ is the $F_c$.

$$F_a = F_c$$

b) What is period of moon (in days) in its orbit around Earth?

One cycle time

$$\frac{1}{V^3} = t$$

$$t = \frac{2\pi \times 3.82 \times 10^8}{1.02 \times 10^3} = 2353114.5 \text{ seconds}$$

$$2353114.5 = 27.2 \text{ days}$$
3) Gravitational Field Strength

m is located at distance $r$ from M.
Gravitational field strength $g$ is force per unit mass on m due to presence of M.

$$g = \frac{F}{m} \text{  Gravitational Field strength [N/kg]}$$

- $F = m a$
- $N = kg \times m/s^2$
- $N = kg \times m/s^2$
- $N \cdot kg^{-1} = m/s^2$

On Earth's surface: $g = 9.8 m/s^2$

$g$ is none other than the gravitational field strength.

4) Solving problems involving gravitational field strength

Suppose $m$ is on surface of $M$ with radius $R$.

From universal law of gravitation, weight of $m$ is equal to its attraction to the planet's mass $M$ and equals $F = \frac{G m M}{R^2}$

$$mg = \frac{G M m}{R^2}$$

- $g = \frac{G M}{R^2}$  Gravitational field strength at surface of a planet of mass $M$ and radius $R$.

The same derivation works for any $r$.

- $g = \frac{G M}{r^2}$  Gravitational field strength at distance $r$ from center of a planet of mass $M$. 
Practice: Mass of Earth is \( M = 5.98 \times 10^{24} \text{ kg} \) and the radius of Earth is \( 6.37 \times 10^6 \text{ m} \). Find gravitational field strength at Earth's surface, and at a distance of one Earth radius above its surface.

\[
g = \frac{GM}{r^2}
\]

\[
g_1 = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.37 \times 10^6)^2} = 9.83 \text{ N kg}^{-1} \text{ (m s}^2) \]

\[
g_2 = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(2 \times 6.37 \times 10^6)^2} = 2.46 \text{ N kg}^{-1} \text{ (m s}^2) \]

Practice: Satellite 350 kg is launched from Earth's surface to a height of one Earth radius above the surface. a) Weight at surface and b) at altitude?

\[
g = \frac{GM}{r^2}
\]

\[
g_1 = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(6.37 \times 10^6)^2} = 9.83 \text{ N kg}^{-1} \text{ (m s}^2) \]

\[
W = mg_1 = 350 \times 9.83 = 3.46 \times 10^3 \text{ N}
\]

\[
g_2 = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(2 \times 6.37 \times 10^6)^2} = 2.46 \text{ N kg}^{-1} \text{ (m s}^2) \]

\[
W = mg_2 = 350 \times 2.46 = 866 \times 10^2 \text{ N}
\]

\[
W = 1.29 \times 10^3 \text{ N}
\]

9) Gravitational field strength @ again)

\[ F = \frac{GMm}{r^2} \quad g = \frac{GM}{r^2} \]

(\text{Force - action at a distance})

(\text{Field - local curvature of space})

(\text{Two masses; and force is the result of their interaction.})

(\text{Just one mass that sets up the local field in the space surrounding it.})

Field view of universe is preferred over the force view. No action at a distance. Force signal will be slightly delayed in telling orbital mass when to turn. End result would be an expanding spiral motion. We do not observe planets leaving their orbits as they travel around the sun. Thus, action at a distance doesn't work. If we believe special relativity, current evidence points to correctness of special relativity. Field view eliminates the need for long distance signaling between 2 masses. It disturbs spacetime about one mass.
- We look at concentration of field lines to know how strong the field is at a particular point in the vicinity of a mass.
- The closer together the field lines, the stronger the field.
- Shaded portion lines are closer together than striped portion.
- Shaded field is thus stronger than striped field.

b) Solving problems involving gravitational field strength - (again)

Q: Find gravitational field strength at a point between the Earth and the moon that is right between their centers.

\[ r = \frac{3.82 \times 10^8}{2} \]

\[ E_M = \frac{6.67 \times 10^{-11} \times 7.36 \times 10^{22}}{3.9 (1.91 \times 10^8)^2} \]

\[ E_E = \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(1.91 \times 10^8)^2} \]

\[ E_M = 0.000135 \text{ Nkg}^{-1} \]

\[ E_E = 0.0109 \text{ Nkg}^{-1} \]

\[ E = 0.0109 - 0.000135 = 0.009765 \text{ N}, \text{ towards the Earth.} \]

\[ = 1.08 \times 10^{-2} \text{ Nkg}^{-1} \]

Practice
Jupiter's gravitational field strength at its surface is 25 Nkg^{-1} while its radius is 7.1 \times 10^7 m. a) Deduce Jupiter's mass. b) Weight of 65 kg man on Jupiter.

a) \[ \frac{25}{6.67 \times 10^{-11}} = \frac{6.67 \times 10^{-11} \times M}{(7.1 \times 10^7)^2} \]

\[ 25 \times (7.1 \times 10^7)^2 = M = 1.89 \times 10^{27} \text{ kg} \]

b) \[ 25 \times 65 = 1625 \text{ N} \]

\[ = 1600 \text{ N} \]
Practice: Two spheres of equal mass & different radii are held a distance apart. The gravitational field strength is measured on the line joining the 2 masses at position x, which varies. Which graph shows the variation of g with x correctly?

At the surfaces, both masses have their biggest strengths E. But, \( m_2 \) has higher radius, so its E has surface is lower than \( m_1 \). a & b are wrong.

Somewhere closer to \( m_2 \), \( r_2 + x^2 = r_2 + (d-x) \), so E is 0. C is wrong. D is correct.

1) Solving problems involving orbital period.

Kepler's law - period \( T \) of an object in a circular orbit about a body of mass M is given by \( T^2 = \left[ (4\pi^2/\text{GM}) \right] r^3 \). Derive it.

\[ F_c = m_1 a_c, \text{ in circular orbit.} \]

\[ F_c = \frac{\text{GMm}}{r^2}, \text{ from newton's law of gravitation} \]

\[ a_c = \frac{4\pi^2 r}{T^2}, \text{ from G-1} \]

\[ \frac{\text{GMm}}{r^2} = \frac{\text{GM}}{r^2} \]

(Not in data booklet!) Memorize!
Example: A satellite in geosynchronous orbit takes 24 hours to orbit the Earth. So, it can be above some point of Earth at all times, if desired. Find the necessary radius & express it in terms of earth radii. \( R_E = 6.37 \times 10^6 \) m.

Kepler's law:

\[
T^2 = \left( \frac{4\pi^2}{GM} \right) r^3
\]

\[
86400^2 = \frac{4\pi^2 r^3}{GM}
\]

\[
86400^2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{-24} = 4\pi^2 r^3
\]

\[
86400^2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{-24} = \frac{r^3}{4\pi^2}
\]

\[
r = \sqrt[3]{\frac{86400^2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{-24}}{4\pi^2}}
\]

\[
r = 4.23 \times 10^7 \text{ m}
\]

4.23 \times 10^7 \text{ m} = 6.63 \text{ times Earth's radius.}

6.37 \times 10^6 \text{ m} = 6.63 R_E

Kepler Third Law:

\[
T^2 = \left( \frac{4\pi^2}{GM} \right) r^3
\]

Not in data booklet!

Initially, this was just \( T^2 \propto r^3 \). Newton's law was needed to get the proportionality constant.

Example: Rings of Saturn are made of rocky particles that orbit the planet. Period \( T \) of each particle depends on distance \( r \) from the centre of Saturn. \( T \) is proportional to \( r^n \).

What is \( n \)?

\[
T^2 = \left( \frac{4\pi^2}{GM} \right) r^3
\]

\[
\sqrt{T^2} = \sqrt{\frac{4\pi^2}{GM} (r^3)}
\]

\[
T = \left( \frac{4\pi^2}{GM} \right)^{\frac{1}{2}} r^{\frac{3}{2}}
\]

\[
n = \frac{3}{2} = 1.5 \quad T \propto r^{1.5}
\]
(2) Two satellites $X$ & $Y$, move in circular orbits about Earth. Orbital period of satellite $X$ is eight times of satellite $Y$.

Find ratio $\frac{\text{orbital radius of satellite } X}{\text{orbital radius of satellite } Y}$

\[
T_x^2 = \left(\frac{4\pi^2}{GM}\right)^3 \quad T_y^2 = \left(\frac{4\pi^2}{GM}\right)^3
\]

\[
x = 64y
\]

64 times period of $Y$ when squared \[T_x^2 = 64T_y^2\]

\[
T_x^2 = 64
\]

\[
1 = \frac{\frac{4\pi^2}{GM} r_x^3}{\frac{4\pi^2}{GM} r_y^3} \quad 64 = \frac{r_x^3}{r_y^3} = \frac{T_x^2}{T_y^2}
\]

\[
(64)^{\frac{1}{3}} = \left(\frac{r_x}{r_y}\right)^{\frac{1}{3}}
\]

\[
4 = \frac{r_x}{r_y}
\]

8) Solving problems involving gravitational field.

If Dobson is in an elevator, and he drops a ball. Accelerate downward is 10 m/s².

But, what about when the elevator is accelerating upwards at 2 m/s²? What is that?

Since, elevator moves 2 m/s² upwards to meet the ball, and ball goes down at 10 m/s².

Dobson would observe an acceleration of 12 m/s².

If elevator was going down at 2 m/s², then he will observe ball to be accelerating at 8 m/s².

If elevator goes at 10 m/s² down, 0 acceleration observed for ball. He thinks it is "weightless".
Astronauts feel weightlessness because they and the spacecraft accelerate at the same value, \( a_c = g \). They fall together and appear weightless.

In deep space, we are truly weightless.

\( r \) is so large for every \( m \) in \( \frac{GMm}{r^2} \) that Force \( F \), the force of gravity, is almost zero.

**Practice**

Satellite of mass \( m \) and speed \( v \) orbits the Earth at a distance \( r \) from the center of the Earth. The gravitational field strength due to Earth is:

\[
F = mg = \frac{mv^2}{r} = \frac{GMM}{r^2}
\]

Cancel \( m \)

\[
E = \frac{GM}{r}
\]

Acceleration

Gravitational field strength is \( \frac{v^2}{r} \)

**Practice**

A satellite of mass \( m \) orbits a planet of mass \( M \) and radius \( R \) as shown. Radius of circular orbit is \( x \). The planet can be as point mass with mass concentrated at center.

a) Deduce that linear speed \( v \) of satellite in its orbit is \( v = \sqrt{\frac{GM}{x}} \).

\[
F_c = \frac{mv^2}{r} = \frac{GMM}{x^2}
\]

b) Derive expression in \( m, G, M, \) and \( x \), for kinetic energy of satellite:

\[
K_c = \frac{1}{2} \frac{mv^2}{x^2}
\]

\[
\frac{mv^2}{x^2} = \frac{GMM}{x^2} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{x}}
\]
Solving problems with gravitational force

1. A binary star consists of 2 stars that each follow circular orbits about a fixed point.

2. Consider the force acting on one of the stars, deducing orbital period T, given by:
   \[ T = \sqrt{\frac{4\pi^2}{G(M_1 + M_2)}} \]
   where \( M_1 \) and \( M_2 \) are the masses of the stars, and R is the distance between them.

3. The gravitational force between the stars is given by:
   \[ F = \frac{G M_1 M_2}{R^2} \]
   where \( G \) is the gravitational constant, \( M_1 \) and \( M_2 \) are the masses, and R is the distance between them.

4. The stars have some orbital period T, can be considered as real masses, and are concentrated at its center. The stars, M_1 and M_2, orbit at distances R_1 and R_2, respectively, from point P.

5. The orbital period for a star is given by:
   \[ T = \sqrt{\frac{4\pi^2 R^3}{GM}} \]
   where R is the distance from the center of the system and M is the total mass of the system.

6. Explain when star has the larger orbit & which star has the larger orbit and why R > R_2 compared to R_1.

7. Find the orbit r of the stars, orbit R, orbit R_2, orbit R_1.

8. Discuss the above problem, show that m_2 is closer to point P than m_1. Using (r) state & orbital period and distance.

9. The gravitational force is given by:
   \[ F = \frac{G M_1 M_2}{R^2} \]
   where \( F \) is the force, \( G \) is the gravitational constant, \( M_1 \) and \( M_2 \) are the masses, and R is the distance between them.

10. The orbital period for a star is given by:
    \[ T = \sqrt{\frac{4\pi^2 R^3}{GM}} \]
    where R is the distance from the center of the system and M is the total mass of the system.

11. The total mass of the system is given by:
    \[ M = M_1 + M_2 \]
    where \( M_1 \) and \( M_2 \) are the masses of the stars.
1) \[ \begin{align*}
X, \quad a_x &= a \\
Y, \quad a_y &= r = R \\
V &= v \\
v &= \omega r \\
v &= \omega R \\
&= \frac{v}{r} = \frac{R}{R} \\
a_x &= \omega^2 r \\
a_y &= \omega^2 R
\end{align*} \]

D. Speed \( v = 2v \), Acceleration = 2a

2) Car moves along a path that is part of a vertical circle of radius \( R \). Speed of car at highest point is \( v \). What is the max value of \( v \) so that the car does not lose contact with road?

\[ \begin{align*}
&\text{Normal Force:} \\
&\frac{mv^2}{r} = mg - N \quad \text{(normal force)} \\
&\frac{mv^2}{r} = mg - N \\
mg - \frac{mv^2}{r} = N
\end{align*} \]

If car loses contact, \( mg \) & \( mg \) Centripetal force is not enough to keep it on track. It is when \( mg = N \), and resultant vertical force is 0.

\[ \begin{align*}
&\frac{mv^2}{r} = mg \\
v^2 = \frac{mg}{r} \\
v = \sqrt{\frac{mg}{r}} \\
B. v_{\text{max}} = \sqrt{gr} \\
&F = \frac{kq_1 q_2}{r^2} \\
a = \frac{kq_1^2}{r^2}
\end{align*} \]

<table>
<thead>
<tr>
<th></th>
<th>Magnitude</th>
<th>Direction</th>
<th>( \frac{d^2}{dt^2} = \frac{q_1}{r^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Constant</td>
<td>Changing</td>
<td>( a ) is changing in</td>
</tr>
<tr>
<td>B</td>
<td>Constant</td>
<td>Constant</td>
<td>direction. Magnitude is 0.</td>
</tr>
<tr>
<td>C</td>
<td>Changing</td>
<td>Changing</td>
<td>So, magnitude of ( q ) is 0</td>
</tr>
<tr>
<td>D</td>
<td>Changing</td>
<td>Constant</td>
<td>constant. Direction changing.</td>
</tr>
</tbody>
</table>
A probe orbits a planet in circular orbit of radius $r$. Period $T$, what is the mass of the planet?

\[ T = \frac{2\pi}{\omega} \]
\[ \omega = \frac{2\pi}{T} \]
\[ V = \omega r \]
\[ F = ma = \frac{GMm}{r^2} \]
\[ \frac{GM}{r^2} = \frac{4\pi^2r}{T^2} \]
\[ \frac{GM}{r^2} = \frac{4\pi^2r^3}{T^2} \]
\[ M = \frac{4\pi^2r^3}{T^2} \]

Option A.

Mass of X is 4 times mass of Y.

Net gravitational field strength along the dotted line a distance $r$ from centre of X is $g$. Positive $g$ means field is towards right.

Which graph shows the variation $g$ with $r/d$?

A

B

C

D

Solved on next page.
As \( r/d \) increases, \( r \) increases

\[ m_x = \frac{1}{y} \quad m_y = y \]

\( +y \) means towards right.

\( r_1 < r_2 < r_3 \)

\[ \frac{GxM}{r_1^2} = \frac{GM}{(d-r)^2}, \quad \frac{GyM}{r_2^2} = \frac{6M}{(d-r)^2} \]

\( GM \) is smaller than \( \frac{GM}{r_1^2} \). \( GM \) is bigger than \( \frac{GM}{(d-r)^2} \)

It decreases as \( r \) increases. So, decreased with \( \frac{r}{d} \).

C & D options are out.

\[ 0 = \frac{GxM}{r_1^2} - \frac{GM}{(d-r)^2} \]

\[ \frac{GM}{(d-r)^2} \]

\[ \frac{GxM}{r_1^2} = \frac{4GM}{(d-r)^2} \]

\[ \frac{GyM}{r_1^2} = \frac{GM}{(d-r)^2} \]

\[ \frac{2}{r_1^2} > (d-r)^2 \]

\[ r > d-r \]

\[ 2r > d \]

\[ r^2 = 4(d-r)^2 \]

\[ r^2 = 4(d^2 + r^2 - 2dr) \]

\[ 4d^2 + 4r^2 - 8dr = 0 \]

\[ (2d)(2d) \]

Book Says answer is C; I don't know.
A planet has 3 times the mass & 3 times the radius of Earth. If $g_s$ at the surface of Earth is $g$, what is $g_s$ at the surface of the planet?

\[ g = \frac{GM}{r^2} \]
\[ a = \frac{3GM}{(3r)^2} - \frac{GM}{9r^2} - \frac{GM}{3r^2} \]
\[ a = \frac{1}{9} \cdot \frac{g}{3} \]

\[ \text{A) } g \quad \text{B) } \frac{g}{3} \quad \text{C) } \frac{g}{9} \quad \text{D) } \frac{g}{27} \]

What is the ratio $\frac{v_x}{v_y}$ of orbital speeds of the 2 satellites?

\[ v_x = \omega R \quad v_y = 2\omega R \]
\[ \omega = \frac{2\pi}{T} \quad \omega = \frac{2\pi}{T} \]

\[ \frac{v_x}{v_y} = \frac{1}{2} \]

\[ m \left( \frac{v^2}{r} \right) = \frac{GMm}{r^2} \]
\[ x = \frac{GM}{r^2} \quad v^2 = \frac{GM}{r} \quad y = \frac{GM}{r} \]
\[ v^2 \propto \frac{1}{r} \]
\[ v_x = \sqrt{\frac{GM}{R}} \quad v_y = \sqrt{\frac{GM}{2R}} \]
\[ \frac{v_x}{v_y} = \frac{\sqrt{\frac{GM}{R}}}{\sqrt{\frac{GM}{2R}}} = \sqrt{2} \]

\[ \text{D) } \sqrt{2} \]

What is correct about a probe that orbits the Earth in a circular orbit at constant speed?

A. Acceleration is constant
B. Velocity is constant
C. Kinetic energy is constant
D. Momentum is constant
9) Two stars have masses 4M and M.

At which point is magnitude of combined gravitational field least?

A \quad B \quad C \quad D \quad \text{At C.}

10) \quad F = \frac{Gm^2}{r^2} \quad \text{to calculate gravitational force between two uniform cubes with side a & mass M.}

Result will be:

A \quad \text{Correct}
B \quad \text{Approximately correct}
C \quad \text{Approximately correct only if } a = r
D \quad \text{Approximately correct only if } a \ll r, \text{ volume can't be too much? idk?}